

Basic Concepts: An equation which has derivative (derivatives) of the dependent variable (like y) with respect to independent variable (like x), is called a Differential Eqⁿ.

e.g. $x \left(\frac{dy}{dx} \right) + y = 0$ ✓

e.g. $2 \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^3 = 0$ ✓

e.g. $\frac{d^2y}{dx^2} = 5$ ✓

Ordinary
Diff. Eqⁿ.
Only one
Indep.
Variable

Note: $\frac{dy}{dx} = y' = y_1$

$\frac{d^3y}{dx^3} = y''' = y_3$

$\frac{d^2y}{dx^2} = y'' = y_2$

$\frac{d^ny}{dx^n} = y_n$

Order of a differential Equation (कोटि) = (Order of Highest order derivative)

e.g. $\left(\frac{dy}{dx} \right) = e^x \rightarrow$ ~~order = 1~~ order = 1

e.g. $\left(\frac{d^2y}{dx^2} \right) = -y \rightarrow$ order = 2

e.g. $\left(\frac{d^3y}{dx^3} \right) = -x^2 \left(\frac{d^2y}{dx^2} \right)^3 \rightarrow$ order = 3

Degree of a Differential Equation: (घात)

→ इसके लिए equation, derivatives में Polynomial होना प्यारी है।

→ Derivatives में Polynomial format होने के बाद ~~Highest~~ Highest order derivative की Highest power = Degree.

e.g. $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0$ ✓
 Order = 3
 Degree = 1 ✓

e.g. $\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin\sqrt{y} = 0$ ✓
 Order = 1
 Degree = 2

e.g. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$ } Not in Polynomial format
 Order = 1
 Degree = Not Defined

e.g. $\left(\frac{dy}{dx}\right)' + \sin x = 0$ ✓
 Order = 1
 Degree = 1

e.g. $x + \sin\left(\frac{dy}{dx}\right) = 0 \Rightarrow \sin\frac{dy}{dx} = -x$
 $\Rightarrow \frac{dy}{dx} = \sin^{-1}(-x)$
 Order = 1
 Degree = 1

e.g. $y''' + y^2 + e^{y'} = 0$
 Order = 3
 Degree = Not Defined

e.g. $y_2 + \sqrt{y_1} = 0$ ✗
 $y_2 = -\sqrt{y_1}$
 $(y_2)^2 = (y_1)$ ✓
 Order = 2
 Degree = 2

Exercise 9.1

Determine order and degree (if defined) of differential equations.

$$\boxed{\text{Q.1}} \quad \frac{d^4 y}{dx^4} + \sin(y''') = 0$$

order = 4

Degree = Not Defined

$$\Rightarrow y_4 + \sin(y_3) = 0$$

$$\Rightarrow \sin(y_3) = -y_4$$

$$\Rightarrow (y_3) = \sin^{-1}(-y_4)$$

Polynomial
format

Q.2

~~y'~~

$$y' + 5y = 0$$

\Rightarrow

$$\frac{dy}{dx} + 5y = 0$$

order = 1

Degree = 1

Q.3

$$\left(\frac{ds}{dt}\right)^4 + 3s \frac{d^2 s}{dt^2} = 0$$

order = 2

Degree = 1

Q.4

$$\left(\frac{d^2 y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

order = 2

Degree = Not
Defined.

Q.5 $\left(\frac{d^2y}{dx^2}\right) = \cos 3x + \sin 3x$ Order = 2
 Degree = 1

Q.6 $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ Order = 3
 Degree = 2

Q.7 $y''' + 2y'' + y' = 0 \rightarrow$ Order = 3
 Degree = 1

Q.8 $y' + y = e^x \rightarrow$ Order = 1
 Degree = 1

Q.9 $y'' + (y')^2 + 2y = 0 \rightarrow$ Order = 2
 Degree = 1
 H.O.D.

Q.10 $y'' + 2y' + \sin y = 0 \rightarrow$ Order = 2
 Degree = 1

Q.11 The degree of the differential equation

$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is \rightarrow (A) 3 (B) 2
 (C) 1 (D) Not Defined
 Derivatives Polynomial \rightarrow Not a Polynomial in Deri.

Q.12 The order of the differential equation

$2x^2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0$ is —
 (A) 2 (B) 1 (C) 0 (D) Not Defined.

General & Particular Solutions of a D.E.

For Example.

$$\frac{dy}{dx} = y \leftarrow \text{Differential Equation}$$

General Solution $y = k e^x$ $k \in \mathbb{R}$

Arbitrary Constants (स्वच्छ अचर)

Particular Solutions

$$\begin{cases} y = e^x \checkmark \\ y = 2e^x \checkmark \\ y = 3e^x \checkmark \\ \vdots \end{cases}$$

function

Class 11th Example

$\sin x = 0$

→ General Solution

$x = n\pi, n \in \mathbb{I}$

→ Particular Solutions

$$\begin{cases} x = 0 \\ x = \pi \\ x = 2\pi \\ \vdots \\ \vdots \end{cases}$$

Definition. General Solution → Solution which have Arbitrary Constants (a, b, c, c₁, ...)

Particular Solutions → general solutions से मिले हुए ऐसे Solutions जिनमें Arbitrary constants की जगह कुछ रास value (number) हो।
(Free from arbitrary constants)

e.g. Verify that the function $y = e^{-3x}$ is a solution of the differential equations $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Solution: $y = e^{-3x}$ ✓

$$y' = \frac{dy}{dx} = -3e^{-3x} \quad \checkmark$$

$$y'' = \frac{d^2y}{dx^2} = 9e^{-3x} \quad \checkmark$$

$$\begin{aligned} \text{LHS} &= y'' + y' - 6y \\ &= 9e^{-3x} + (-3e^{-3x}) - 6(e^{-3x}) \\ &= 0 = \text{RHS} \end{aligned}$$

LHS = RHS

General Solution

e.g. Verify that the function $y = a \cos x + b \sin x$, where $a, b \in \mathbb{R}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solⁿ: $y = a \cos x + b \sin x$ ✓

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

$$\frac{d^2y}{dx^2} = -a \cos x - b \sin x \quad \checkmark$$

$a, b \rightarrow$ Arb. Const.

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + y \\ &= -a \cos x - b \sin x + a \cos x + b \sin x \\ &= 0 = \text{RHS} \end{aligned}$$

Note: Order of Differential Equation = Number of Arbitrary Constants in General Solution

order = order of $\textcircled{2}$ = $\textcircled{2}$
highest order derivative

Exercise 9.2

LHS = RHS

Verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

Q.1 $y = e^x + 1$: $y'' - y' = 0$ (D.E.) \rightarrow **LHS = RHS**

$y' = e^x$
 $y'' = e^x$

LHS = $y'' - y'$
 \uparrow $= e^x - e^x = 0 =$ RHS \uparrow

verified

Q.2 $y = x^2 + 2x + c$: $y' - 2x - 2 = 0$

$\Rightarrow y' = 2x + 2$

\Rightarrow $y' - 2x - 2 = 0$
 LHS = RHS

Q.3 $y = \cos x + c$: $y' + \sin x = 0$

$\Rightarrow y' = -\sin x$

LHS = $y' + \sin x$
 $= -\cancel{\sin x} + \cancel{\sin x} = 0 =$ RHS

verified

Q.4 $y = \sqrt{1+x^2}$: $y' = \frac{xy}{1+x^2}$

\Rightarrow $y' = \frac{1}{\sqrt{1+x^2}} \times (2x)$

$\Rightarrow y' = \frac{2x}{\sqrt{1+x^2}}$

LHS = $y' = \frac{x}{\sqrt{1+x^2}}$

RHS = $\frac{x(y)}{1+x^2} = \frac{x \sqrt{1+x^2}}{(1+x^2)^1} = \frac{x}{\sqrt{1+x^2}}$

LHS = RHS

Q.9 $x+y = \tan^{-1}y$: $y^2 \frac{dy}{dx} + y^2 + 1 = 0$

Diff. w.r.t (x)
 $\Rightarrow 1 + y' = \frac{1}{1+y^2} \cdot y'$

$\Rightarrow (1+y')(1+y^2) = y'$

$\Rightarrow 1+y^2 + \cancel{y'} + \cancel{y'} \cdot y^2 = \cancel{y'}$

$y^2 \cdot y' + y^2 + 1 = 0$

Q.10 $y = \sqrt{a^2 - x^2}$, $x+y \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{1 \cdot (0 - 2x)}{2\sqrt{a^2 - x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}} = \frac{-x}{y}$

LHS = $x+y \cdot \left(\frac{dy}{dx}\right)$
 $= x+y \left(\frac{-x}{y}\right)$
 $= x-x = 0 = \text{RHS}$

Q.11 The number of arbitrary constants in the general solⁿ of a differential equation of fourth order are:

- (A) 0 (B) 2 (C) 3 (D) 4

order of D.E. = No. of Arbitrary Constants in General Solution.

Q.12 The number of arbitrary constants in the particular solution of a differential equation of third order are —

- (A) 3 (B) 2 (C) 1 (D) = 0

Particular Solutions \rightarrow No Arbitrary Constants.

Formation of Differential Equation :->

e.g. $x, y, \frac{dy}{dx}, y', y'' \dots$

General Solution (given) Eqⁿ \longrightarrow Differential Equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

\longrightarrow

$$xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \left(\frac{dy}{dx} \right) = 0$$

Arbitrary Constants ($\frac{4}{3+2\sqrt{2}}$)

Note: Order of differential Equation = Number of arbitrary Constants ★

Steps to Form a Differential Equation ->

- ① Given general solution को उतनी बार differentiate करके रख लो, जितने उसमें Arbitrary Constants हो।
- ② जितनी भी equations अब आपके सामने हों, उनसे Arbitrary Constants को हरा लो। Done!!!

e.g. Form the differential equation representing the family of curves $y = a \sin(x+b)$, where a, b are arbitrary Constants.

Ans. No. of Arb. Constants = 2 $\leftarrow \begin{matrix} a \\ b \end{matrix}$ (order = 2)

$$y = a \sin(x+b) \quad \text{--- (1)}$$

$$y' = a \cos(x+b) \quad \text{--- (2)}$$

$$y'' = -a \sin(x+b) \quad \text{--- (3)}$$

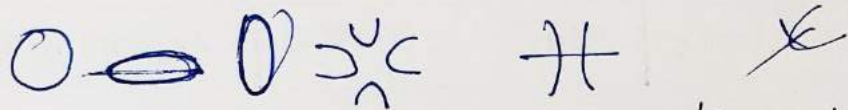
By eqⁿ (1) & (3) :->

$y'' = -(y)$

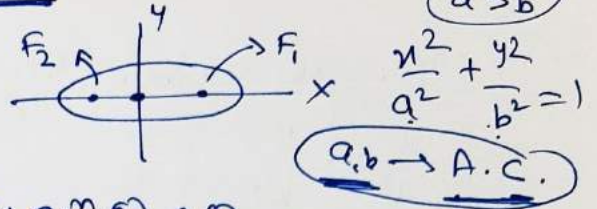
\longrightarrow

$\frac{d^2y}{dx^2} + y = 0$

$y'' + y = 0$



e.g. Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.



Ans. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ — (1)

Diff. $\Rightarrow \frac{2x}{a^2} + \frac{2y \cdot y'}{b^2} = 0$

$\Rightarrow \frac{x}{a^2} + \frac{y \cdot y'}{b^2} = 0$ — (2)

Diff. $\Rightarrow \frac{1}{a^2} + \frac{y' \cdot y'}{b^2} + \frac{y \cdot y''}{b^2} = 0$

$\Rightarrow \frac{1}{a^2} = -\frac{(y')^2}{b^2} - \frac{y \cdot y''}{b^2}$ — (3)

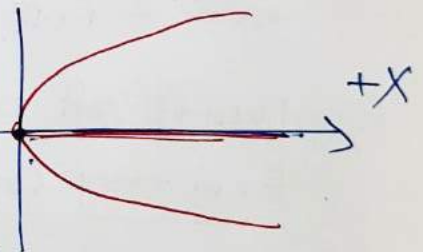
By eqⁿ (2) & (3): $\Rightarrow x \left(-\frac{(y')^2}{b^2} - \frac{y \cdot y''}{b^2} \right) + \frac{y \cdot y'}{b^2} = 0$
 $\Rightarrow (-x(y')^2 - xy \cdot y'' + yy') = 0$
 $\Rightarrow xy \cdot y'' + x(y')^2 - yy' = 0$

e.g. Form the differential eqⁿ representing the family of Parabolas having vertex at origin and axis along positive direction of x-axis.

Standard Form: $y^2 = 4ax$

a = arbitrary constant

order = 1



$y^2 = 4ax$ — (1)

Diff. $2y \cdot y' = 4a$ — (2)

by eqⁿ (1) : $\frac{y^2 \cdot y'}{2y \cdot y'} = \frac{4ax}{4a}$

$\Rightarrow \frac{y}{2y'} = x \Rightarrow y = 2xy'$

$y = 2x \cdot \frac{dy}{dx}$

Exercise 9.3

(Formation of Differential Equation)

Form a differential equation:

[Q.1] $\frac{x}{a} + \frac{y}{b} = 1$ — ① No. of Arbitrary Constants = 2 $\begin{matrix} a \\ b \end{matrix}$

Diff w.r.t. 'x'

$$\Rightarrow \frac{1}{a} + \frac{y'}{b} = 0 \text{ — ②}$$

Differentiation = 2

order = 2

Again Diff.

$$\Rightarrow 0 + \frac{y''}{b} = 0 \Rightarrow \boxed{y'' = 0} \text{ Diff. Equation.}$$

[Q.2] $y^2 = a(b^2 - x^2)$ — ① No. of Arb. Constants = 2 $\begin{matrix} a \\ b \end{matrix}$

$$\Rightarrow y^2 = ab^2 - ax^2$$

Diff. = 2

by Diff. w.r.t. (x) \rightarrow

$$\Rightarrow 2y \cdot y' = 0 - 2ax$$

$$\Rightarrow \boxed{y \cdot y' = -ax} \text{ — ②}$$

Again by Diff. w.r.t. (x)

$$\Rightarrow \boxed{y' \cdot y' + y \cdot y'' = -a} \text{ — ③}$$

By eqⁿ ② & ③ \rightarrow

$$\Rightarrow y \cdot y' = ((y')^2 + y \cdot y'') x$$

$$\Rightarrow y \cdot y' = x y \cdot y'' + x (y')^2$$

$$\Rightarrow \boxed{x y y'' + x (y')^2 - y y' = 0}$$

Q3 $y = ae^{3x} + be^{-2x}$ — (1)

Arb = (2) $\begin{cases} a \\ b \end{cases}$

$y' = 3ae^{3x} - 2be^{-2x}$ — (2)

$\begin{cases} a \\ b \end{cases}$

$y'' = 9ae^{3x} + 4be^{-2x}$ — (3)

Elimination

$2y = 2ae^{3x} + 2be^{-2x}$ — (1)
 $+ y' = 3ae^{3x} - 2be^{-2x}$ — (2)

$3y = 3ae^{3x} + 3be^{-2x}$ — (1)
 $y' = 3ae^{3x} - 2be^{-2x}$ — (2)

$2y + y' = 5ae^{3x}$

$3y - y' = 5be^{-2x}$

$\frac{2y + y'}{5} = ae^{3x}$

$\frac{3y - y'}{5} = be^{-2x}$

by putting the values of ae^{3x} & be^{-2x} in eqn (3) \Rightarrow

$\Rightarrow y'' = 9\left(\frac{2y + y'}{5}\right) + 4\left(\frac{3y - y'}{5}\right)$

$\Rightarrow y'' = \frac{18y + 9y' + 12y - 4y'}{5}$

$\Rightarrow y'' = \frac{30y + 5y'}{5} \Rightarrow y'' = 6y + y'$

$\Rightarrow y'' - y' - 6y = 0$ ✓

Q.4 $y = e^{2x}(a+bx)$ — (1) No. of Arbitrary Constants = (2)

$\Rightarrow y' = 2e^{2x}(a+bx) + e^{2x}(0+b \cdot 1)$

$\Rightarrow y' = 2y + be^{2x}$

By eqn (2) / (3)

$\Rightarrow y' - 2y = be^{2x}$ — (2)

$\frac{y' - 2y}{y'' - 2y'} = \frac{be^{2x}}{2be^{2x}}$

Diff. $\Rightarrow y'' - 2y' = 2be^{2x}$ — (3)

Cross multiply

$$\Rightarrow 2y' - 4y = (y'') - 2y'$$

$$\Rightarrow 0 = y'' - 2y' - 2y' + 4y$$

$$\Rightarrow \boxed{y'' - 4y' + 4y = 0} \quad \checkmark$$

$$\boxed{\text{Q.5}} \quad y = e^x (a \cos x + b \sin x) \quad \text{--- (1)}$$

$$\text{As } b = 2 \begin{cases} 9 \\ - \\ b \end{cases}$$

Diff.

$$\Rightarrow y' = \underbrace{e^x (a \cos x + b \sin x)}_y + e^x (-a \sin x + b \cos x)$$

$$\Rightarrow y' = y + e^x (-a \sin x + b \cos x)$$

$$\Rightarrow y' - y = \underbrace{e^x (-a \sin x + b \cos x)}_{\text{again diff.}} \quad \text{--- (2)}$$

$$\Rightarrow y'' - y' = e^x (-a \sin x + b \cos x) + e^x (-a \cos x - b \sin x)$$

$$\Rightarrow y'' - y' = \underbrace{e^x (-a \sin x + b \cos x)}_{\substack{e^{ax} (2) \\ \downarrow \\ y' - y}} - \underbrace{e^x (a \cos x + b \sin x)}_{\substack{\text{by } e^{ax} (1) \\ \downarrow \\ y}}$$

$$\Rightarrow y'' - y' = (y' - y) - (y)$$

$$\Rightarrow y'' - y' = y' - 2y$$

$$\Rightarrow \boxed{y'' - 2y' + 2y = 0} \quad \checkmark$$

Exercise 9.3 [Formation of Differential Equation]

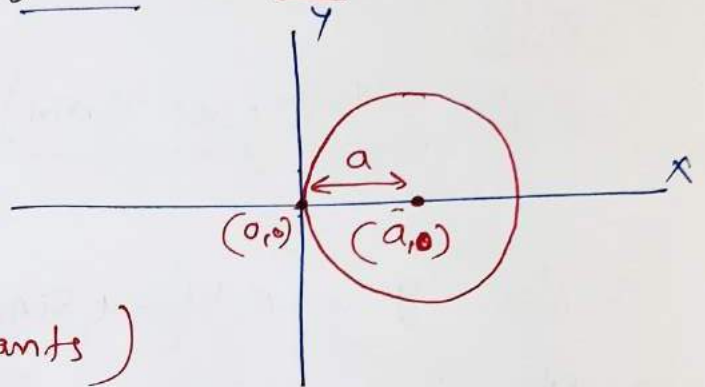
Q.6 Form the differential equation of the ~~ma~~ family of circles touching the y-axis at origin.

Circle

Centre = $(a, 0)$

radius = a (let)

↑
(Arbitrary constants)



→ Eqⁿ, $(x-a)^2 + (y-0)^2 = (a)^2$

⇒ ~~$x^2 + a^2 - 2ax + y^2 = a^2$~~

⇒ $x^2 + y^2 = 2ax$ — (1)

Diff. w.r.t. (x) →

⇒ ~~$2x + 2y \cdot y' = 2a$~~

⇒ $x + yy' = a$ — (2)

No. of Arb. Constants = 1

by eqⁿ (1) & (2) → $x^2 + y^2 = 2(x + yy')$

⇒ $x^2 + y^2 = 2x^2 + 2xyy'$

⇒ $y^2 = x^2 + 2xyy'$

Q.7 Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

Ans.

Standard Eqⁿ. $x^2 = 4ay$

Family of Parabolas $x^2 = 4ay$

D.E. = ?

~~A.C.~~

Arb. Constant = a

$x^2 = 4ay$ (1)

Diff. $2x = 4a \cdot y'$ (2)

by eqⁿ (1)/(2) $\rightarrow \frac{x^2}{2x} = \frac{4ay}{4a \cdot y'} \Rightarrow \frac{x}{2} = \frac{y}{y'}$

$\Rightarrow xy' = 2y \Rightarrow \boxed{xy' - 2y = 0}$ ✓

Q.8 Form the differential eqⁿ. of family of ellipses having foci on y-axis and centre at origin.

Ans.

Standard Form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

No. of Arb. Constants = 2

$a, b \in \mathbb{R}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (1)

Diff. w.r.t. (x)

$\Rightarrow \frac{2x}{a^2} + \frac{2y \cdot y'}{b^2} = 0$

$\Rightarrow \boxed{\frac{x}{a^2} + \frac{y \cdot y'}{b^2} = 0}$ (2)

Again by Diff. w.r.t. (x)

$\frac{1}{a^2} + \frac{(y')^2 + y \cdot y''}{b^2} = 0$

$\Rightarrow \frac{1}{a^2} = - \left[\frac{(y')^2 + y \cdot y''}{b^2} \right]$

(3)

$$\frac{x}{a^2} + \frac{yy'}{b^2} = 0 \quad \text{--- (2)}$$

$$\frac{1}{a^2} = - \frac{(y')^2 + yy''}{b^2} \quad \text{--- (3)}$$

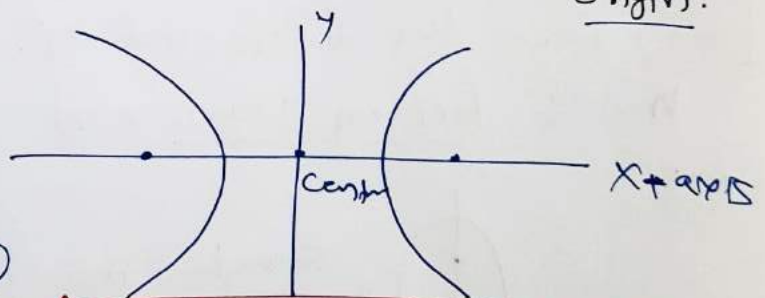
by eqn. (2) & (3) \rightarrow

$$\Rightarrow x \cdot \left(- \frac{(y')^2 + yy''}{b^2} \right) + \frac{yy'}{b^2} = 0$$

$$\Rightarrow \cancel{\frac{xy}{b^2}} - \frac{xy(y')^2 - xyy''}{b^2} + \frac{yy'}{b^2} = 0$$

$$\Rightarrow \boxed{xyy'' + x(y')^2 - yy' = 0} \quad \text{D.F.}$$

Q. 9 Form the differential equation of the family of Hyperbolas having foci on x-axis and centre at origin.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$$

Diff. w.r.t. x

$$\Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2} \cdot y' = 0$$

$$\Rightarrow \frac{x}{a^2} = \frac{y \cdot y'}{b^2} \quad \text{--- (2)}$$

Diff. w.r.t. x

$$\Rightarrow \frac{1}{a^2} = \frac{(y')^2 + y \cdot y''}{b^2} \quad \text{--- (3)}$$

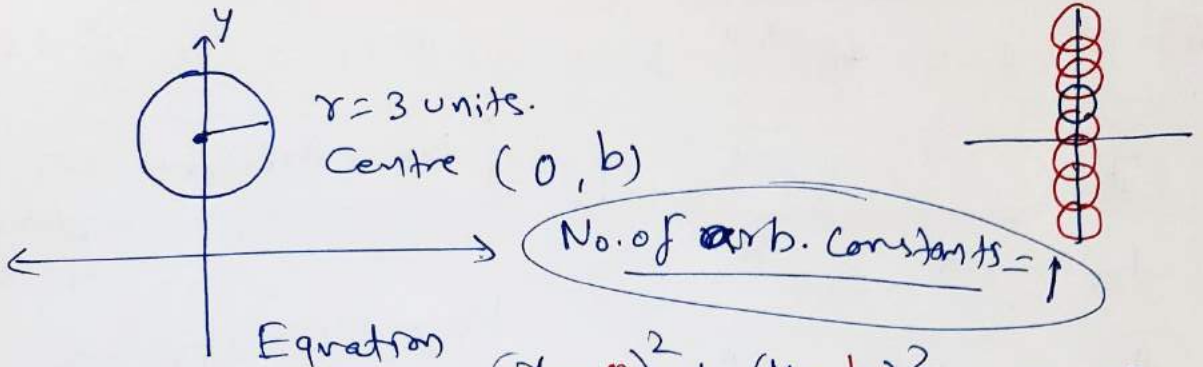
No. of Arb. Const. = 2 \rightarrow a, b

by eqn (2) & (3)

$$\Rightarrow x \left(\frac{(y')^2 + yy''}{b^2} \right) = \frac{yy'}{b^2}$$

$$\Rightarrow \boxed{xyy'' + x(y')^2 - yy' = 0}$$

Q.10 Form the Differential equation of the family of Circles having centre on y-axis and radius 3 units.



Equation

$$(x-0)^2 + (y-b)^2 = (3)^2$$

$$\Rightarrow x^2 + (y-b)^2 = 9 \quad \text{--- (1)}$$

diff. w.r.t. x \rightarrow

$$\Rightarrow 2x + 2(y-b) \cdot y' = 0$$

$$\Rightarrow x + (y-b) \cdot y' = 0 \Rightarrow \underline{(y-b) = \frac{-x}{y'}} \quad \text{--- (2)}$$

by eqn (1) & (2) \rightarrow

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9 \Rightarrow x^2 + \frac{x^2}{(y')^2} = 9$$

$$\Rightarrow (x^2 - 9) + \frac{x^2}{(y')^2} = 0 \Rightarrow \left[\frac{(x^2 - 9) \cdot (y')^2 + x^2}{(y')^2} \right] = 0$$

$$\Rightarrow \underline{(x^2 - 9) \cdot (y')^2 + x^2 = 0}$$

Q.11 Which of the following differential eqⁿ has

$y = c_1 e^x + c_2 e^{-x}$ as the general solution?

- (A) $y'' + y = 0$ (B) $y'' - y = 0$ (C) $y'' + 1 = 0$ (D) $y'' - 1 = 0$

$y = c_1 e^x + c_2 e^{-x}$ — (1)

$y' = c_1 e^x - c_2 e^{-x}$ — (2)

$y'' = c_1 e^x + c_2 e^{-x}$ — (3)

No. of Arb. Const. = 2
 $\swarrow \searrow$
 $c_1 \quad c_2$

by eqⁿ (1) & (3) \rightarrow
 $\Rightarrow y'' = y$
 $(y'' - y = 0)$

Q.12 Which of the following differential equation has $y = x$ as one of its particular solution?

- (A) $y'' - x^2 \cdot y' + xy = x$ (B) $y'' + xy' + xy = x$
~~(C) $y'' - x^2 y' + xy = 0$~~ (D) $y'' + xy' + xy = 0$

(A) \rightarrow (x) ✓

(B) \rightarrow (x) ✓

(C) \rightarrow (1) ✓

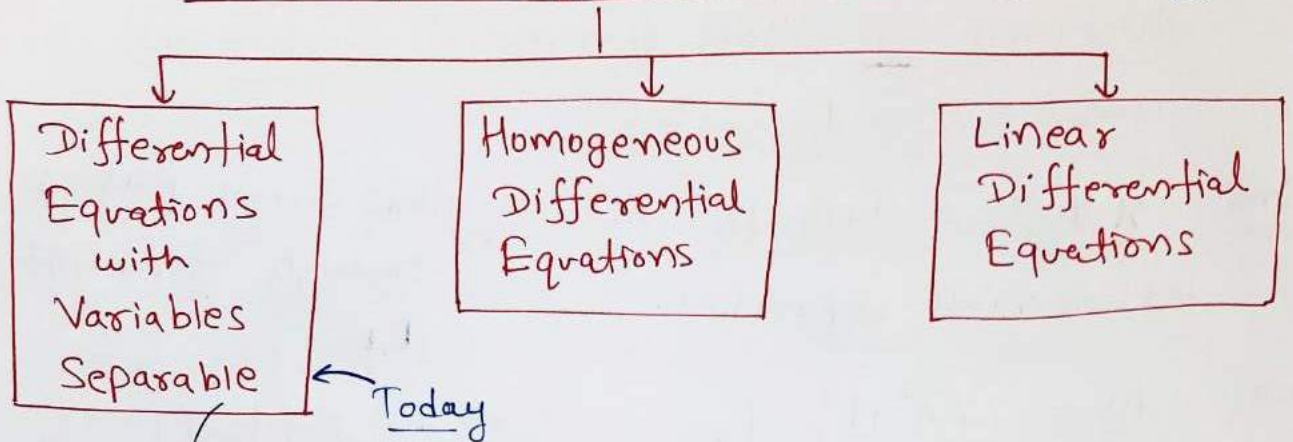
(D) \rightarrow 0 ✓

Check (A) $0 - x^2 \cdot 1 + x^2 = x$
 $0 = x$ ✗

(B) $\rightarrow 0 + x + x^2 = x$
 $x^2 = 0$ ✗

(C) $0 - x^2 + x^2 = 0$
 $\Rightarrow 0 = 0$ ✓

Methods of Solving Differential Equations



Variable Separation method

- First, separate the variables. (x, dx ओले एक तरफ
 y, dy ओले एक तरफ)
- Second, integrate.

e.g. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{(1+y^2)(1+x^2)}$.

$$\int \text{---} \cdot dx$$

Ans by Variable Separation:

$$\Rightarrow \underbrace{(1+y^2)} \cdot dy = \frac{1}{\underbrace{(1+x^2)}} \cdot dx$$

by integrating,

$$\Rightarrow \int (1+y^2) \cdot dy = \int \frac{1}{1+x^2} \cdot dx$$

$$\Rightarrow \boxed{y + \frac{y^3}{3} = \tan^{-1} x + C}$$

Arbitrary Constant

General Solution



e.g. Find the equation of the curve passing through the point (1,1) whose differential equation is

$$x \, dy = (2x^2 + 1) \cdot dx \quad (x \neq 0).$$

Ans. $x \, dy = (2x^2 + 1) \cdot dx$
by variable separation.

$$\Rightarrow dy = \left(\frac{2x^2 + 1}{x} \right) \cdot dx$$

by integrating

$$\Rightarrow \int dy = \int \left(2x + \frac{1}{x} \right) \cdot dx$$

$$\Rightarrow \boxed{y = x^2 + \log|x| + C}$$

Curve

this curve passes through the point (1,1)

$$\Rightarrow 1 = 1 + \log(1) + C$$

$$\Rightarrow \boxed{0 = C}$$

Particular Curve

$$\boxed{y = x^2 + \log|x|}$$

e.g. In a bank, Principal increases continuously at the rate of 5% per year. In how many years ₹1000 double itself.

$$\text{Principal} = P$$

$$\text{Rate of change of Principal} = \frac{d(P)}{dt} = 5\% \text{ of } 'P'$$

$$\Rightarrow \frac{dP}{dt} = \frac{5}{100} \times P \quad \leftarrow \text{Differential eq}^n, \quad (P)$$

$$\Rightarrow \frac{dP}{dt} = \frac{P}{20}$$

$$\Rightarrow \frac{dP}{P} = \frac{dt}{20}$$

Integrate

$$\Rightarrow \int \frac{dP}{P} = \int \frac{dt}{20} \Rightarrow \log|P| = \frac{1}{20}t + C$$

$$P = tve$$

$$\log(P) = \frac{t}{20} + C$$

Present	P = 1000	t = 0 years
Future	P = 2000	T years
	↑	↑
	(P)	(t)

Present: P = 1000

t = 0

put

$$\log(1000) = 0 + C \Rightarrow C = \log(1000)$$

Update

$$\log(P) = \frac{t}{20} + \log(1000)$$

Future { P = 2000 (Double of 1000)
t = T

$$\Rightarrow \log(2000) = \frac{T}{20} + \log(1000)$$

$$\Rightarrow \log(2000) - \log(1000) = \frac{T}{20}$$

$$\Rightarrow \log\left(\frac{2000}{1000}\right) = \frac{T}{20}$$

$$\Rightarrow T = 20 \cdot \log_e(2) \text{ years}$$

$$\Rightarrow T = 20 \times (0.6931)$$

$$T \approx \underline{\underline{13.8 \text{ years}}}$$

$$\begin{aligned} \log_m - \log_n &= \log_m \frac{m}{n} \end{aligned}$$

$$\log_e 2 = \underline{\underline{0.6931}}$$

Exercise 9.4

Variable Separation Method

Q.1 $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

Find the General Solution ($y = ?$)

\Rightarrow (Variable Separation)

$$dy = \frac{1 - \cos x}{1 + \cos x} \cdot dx$$

$$1 - \cos x = 2 \sin^2 \frac{x}{2} \quad \checkmark$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2} \quad \checkmark$$

by integration:

$$\Rightarrow \int dy = \int \frac{1 - \cos x}{1 + \cos x} \cdot dx$$

$$\Rightarrow \int dy = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \cdot dx$$

$$\Rightarrow \int 1 \cdot dy = \int \tan^2 \frac{x}{2} \cdot dx$$

$$\Rightarrow y = \int \left(\sec^2 \frac{x}{2} - 1 \right) \cdot dx$$

$$y = \frac{\tan \frac{x}{2}}{\left(\frac{1}{2}\right)} - x + C$$

$$\Rightarrow \boxed{y = 2 \tan \frac{x}{2} - x + C}$$

General Solⁿ,

Q.2 $\frac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2)$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\Rightarrow \frac{dy}{\sqrt{4 - y^2}} = dx$$

Integration.

$$\Rightarrow \int \frac{dy}{\sqrt{4 - y^2}} = \int dx$$

$$\Rightarrow \sin^{-1} \left(\frac{y}{2} \right) = (x + C)$$

$$\Rightarrow \frac{y}{2} = \sin(x + C)$$

$$\Rightarrow \boxed{y = 2 \sin(x + C)}$$

✓

Q.3 $\frac{dy}{dx} + y = 1 \quad (y \neq 1)$

$\int \frac{1}{x} \cdot dx = \log|x| + C$

$\Rightarrow \frac{dy}{dx} = (1-y)$

V.S. method

$\Rightarrow \frac{dy}{(1-y)} = dx$

integrate:

$\Rightarrow \int \frac{dy}{1-y} = \int dx$

$\Rightarrow -\int \frac{dy}{y-1} = \int dx$

$\Rightarrow \int \frac{dy}{y-1} = -\int dx$

$\Rightarrow \log(y-1) = (-x + C)$

$\Rightarrow (y-1) = e^{-x+C}$

$\Rightarrow y = 1 + e^{-x+C}$

$\Rightarrow y = 1 + e^{-x} \cdot e^C$
 where $e^C = A$
 $\Rightarrow y = 1 + A \cdot e^{-x}$

Q.4 $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

$\Rightarrow \sec^2 x \cdot \tan y \cdot dx = -\sec^2 y \cdot \tan x \cdot dy$

$\Rightarrow \frac{\sec^2 x \cdot dx}{\tan x} = -\frac{\sec^2 y \cdot dy}{\tan y}$

integration \rightarrow

$\Rightarrow \int \frac{\sec^2 x \cdot dx}{\tan x} = -\int \frac{\sec^2 y \cdot dy}{\tan y}$

let $\tan x = t$
 $\sec^2 x \cdot dx = dt$
 $\int \frac{dt}{t}$
 $= \log(t) + C$

$\Rightarrow \log(\tan x) = -\log(\tan y) + C$

$$\Rightarrow \log x = \log y + c$$

$$\Rightarrow \log(\tan x) = -\log(\tan y) + c$$

$$\Rightarrow \log(\tan x) + \log(\tan y) = c$$

$$\Rightarrow \log(\tan x \cdot \tan y) = c$$

$$\Rightarrow \tan x \cdot \tan y = e^c = c_1 \quad \text{New Constant}$$
$$\boxed{\tan x \cdot \tan y = c_1}$$

$$\boxed{\text{Q.5}} \quad (e^x + e^{-x}) \cdot \underline{dy} - (e^x - e^{-x}) \cdot dx = 0$$

$$\Rightarrow (e^x + e^{-x}) \cdot dy = (e^x - e^{-x}) \cdot dx$$

$$\Rightarrow dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot dx$$

integrate \rightarrow

$$\Rightarrow \int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx \rightarrow dt$$

$$\Rightarrow y = \int \frac{dt}{t}$$

$$\Rightarrow y = \log(t) + c$$

$$\Rightarrow \boxed{y = \log(e^x + e^{-x}) + c}$$

let

$$e^x + e^{-x} = t$$
$$(e^x - e^{-x}) \cdot dx = dt$$

Q.6 $\frac{dy}{dx} = \frac{(1+x^2)(1+y^2)}{1+y^2}$

$\Rightarrow \frac{dy}{1+y^2} = (1+x^2) \cdot dx$

integrate \rightarrow

$\Rightarrow \int \frac{dy}{1+y^2} = \int (1+x^2) \cdot dx$

$\Rightarrow \boxed{\tan^{-1}y = x + \frac{x^3}{3} + C}$

Q.7 $y \log y \, dx - x \, dy = 0$

$\Rightarrow y \log y \cdot dx = x \, dy \Rightarrow \frac{dx}{x} = \frac{dy}{y \cdot \log y}$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y \cdot \log y}$

$\Rightarrow \log x = \log(\log y) + C$

$\Rightarrow x = e^{\log(\log y) + C}$

$\Rightarrow x = e^{\log(\log y)} \cdot e^C$

$\Rightarrow x = \log y \cdot e^C$

$\Rightarrow \frac{x}{e^C} = \log y$

$\Rightarrow e^{x/e^C} = y \Rightarrow \boxed{xK = y}$

$\int \frac{dy}{y \cdot \log y} \rightarrow dt$
 $y \cdot \log y \rightarrow t$
 let $\log y = t$
 $\Rightarrow \frac{1}{y} \cdot dy = dt$

 $\int \frac{dt}{t} = \log t + C$
 $= \log(\log y) + C$

$a \log_a b = b$

$\frac{1}{e^C} = K$

$$\boxed{\text{Q.8}} \quad x^5 \frac{dy}{dx} = -y^5 \Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5}$$

by intⁿ,

$$\Rightarrow \int \frac{dy}{y^5} = - \int \frac{dx}{x^5}$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$$

$$\Rightarrow \int y^{-5} \cdot dy = - \int x^{-5} \cdot dx$$

$$\Rightarrow \frac{y^{-5+1}}{-5+1} = - \left(\frac{x^{-5+1}}{-5+1} \right) + C$$

$$\Rightarrow \left\{ \left(\frac{y^{-4}}{-4} \right) = - \left(\frac{x^{-4}}{-4} \right) + C \right\} \times \textcircled{-4}$$

$$\Rightarrow y^{-4} = -x^{-4} - 4C$$

$$\Rightarrow x^{-4} + y^{-4} = \textcircled{-4C} = C_1 = \text{New Constant}$$

$$\boxed{x^{-4} + y^{-4} = C_1}$$

$$\boxed{\text{Q.9}} \quad \frac{dy}{dx} = \sin^{-1}x \Rightarrow \underline{dy} = \underline{\sin^{-1}x \cdot dx}$$

by integration $\rightarrow \int dy = \int \underbrace{\sin^{-1}x}_{\textcircled{\text{I}}} \cdot \underbrace{dx}_{\textcircled{\text{II}}}$

ILATE

$$\boxed{\int \text{I} \cdot \text{II} = \text{I} \int \text{II} - \int (\text{I}' \cdot \text{II})}$$

int. by parts.

$$\Rightarrow y = \sin^{-1}x \cdot (x) - \int \left(\frac{1}{\sqrt{1-x^2}} \cdot x \right) \cdot dx$$

$$\Rightarrow y = x \cdot \sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Subs. $1-x^2 = t$

$$\Rightarrow -2x \cdot dx = dt \Rightarrow \boxed{x dx = \frac{dt}{-2}}$$

$$\Rightarrow y = x \sin^{-1} x - \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$\Rightarrow y = x \sin^{-1} x + \frac{1}{2} (2\sqrt{t}) + C$$

$$\Rightarrow \boxed{y = x \sin^{-1} x + \sqrt{1-x^2} + C}$$

$$\int \frac{1}{\sqrt{t}} \cdot dt$$

$$\int t^{-1/2} \cdot dt$$

$$t^{(-1/2+1)}$$

$$t^{1/2} = 2\sqrt{t}$$

[Q.10] $e^x \tan y \, dx + (1-e^x) \cdot \sec^2 y \cdot dy = 0$

$$\Rightarrow \underline{e^x} \tan y \cdot \underline{dx} = \underline{(e^x - 1)} \cdot \sec^2 y \cdot dy$$

$$\Rightarrow \frac{e^x \cdot dx}{e^x - 1} = \frac{\sec^2 y \cdot dy}{\tan y}$$

integration,

$$\Rightarrow \int \frac{e^x \cdot dx}{e^x - 1} = \int \frac{\sec^2 y \cdot dy}{\tan y}$$

$$\Rightarrow \log(e^x - 1) = \log(\tan y) + C$$

$$\Rightarrow \left(\frac{\log(e^x - 1) - C}{e} \right) = \left(\frac{\log(\tan y)}{e} \right)$$

$$\Rightarrow e^{\log(e^x - 1)} \cdot e^{-C} = \tan y$$

$$\Rightarrow \boxed{(e^x - 1) \cdot K = \tan y} \quad \checkmark$$

Exercise 9.4

Variable Separation Method.

Find a particular solution satisfying the given condition \rightarrow

(Q.11) $(x^3 + x^2 + x + 1) \cdot \frac{dy}{dx} = 2x^2 + x$; $y = 1$ when $x = 0$.

$$\Rightarrow dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1} \cdot dx$$

integration.

$$\Rightarrow \int dy = \int \frac{2x^2 + x}{x^3 + x^2 + x + 1} \cdot dx$$

$$y = \int \frac{2x^2 + x}{x^2(x+1) + 1(x+1)} \cdot dx$$

$$y = \int \frac{2x^2 + x}{(x+1)(x^2+1)} \cdot dx$$

Partial fraction

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Proper Method: $2x^2 + x = A(x^2+1) + (Bx+C) \cdot (x+1)$

Coeff. Comparison. $\Rightarrow (2x^2 + x) = Ax^2 + A + Bx^2 + Bx + Cx + C$

$(x^2) \rightarrow 2 = A + B$ — (1)

$(x) \rightarrow 1 = B + C$ — (2)

Constant $\rightarrow 0 = A + C$ — (3)

$$\begin{aligned} B &= \frac{3}{2} \\ A &= \frac{1}{2} \\ C &= -\frac{1}{2} \end{aligned}$$

$$y = \int \frac{2x^2 + x}{(x+1)(x^2+1)} \cdot dx \Rightarrow y = \int \left(\frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right) \cdot dx$$

$$\Rightarrow y = \frac{1}{2} \int \frac{1}{x+1} \cdot dx + \frac{3}{2 \cdot 2} \int \frac{2x \cdot dx}{x^2+1} - \frac{1}{2} \int \frac{1}{x^2+1} \cdot dx$$

$$\Rightarrow y = \frac{1}{2} \log(x+1) + \frac{3}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1}x + C$$

$$x=0, y=1$$

$$\Rightarrow 1 = \frac{1}{2} \log(1) + \frac{3}{4} \log(1) - \frac{1}{2} \tan^{-1}(0) + C$$

$$1 = C$$

Curve $y = \frac{1}{2} \log(x+1) + \frac{3}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1}(x) + 1$

$$\Rightarrow y = \frac{1}{4} \left\{ 2 \log(x+1) + 3 \cdot \log(x^2+1) \right\} - \frac{\tan^{-1}x}{2} + 1$$

$$n \cdot \log m = \log m^n$$

$$\Rightarrow y = \frac{1}{4} \left\{ \log(x+1)^2 + \log(x^2+1)^3 \right\} - \frac{\tan^{-1}x}{2} + 1$$

$$\log m + \log n = \log mn$$

$$\Rightarrow y = \frac{1}{4} \log \left[(x+1)^2 \cdot (x^2+1)^3 \right] - \frac{\tan^{-1}x}{2} + 1$$

(Q.12) $x(x^2-1) \cdot \frac{dy}{dx} = 1$; $y=0$ when $x=2$

$\Rightarrow dy = \frac{dx}{x(x^2-1)} \rightarrow$ Integrate.

$\Rightarrow \int dy = \int \frac{dx}{x(x^2-1)} = \int \frac{dx}{x(x+1)(x-1)}$
Partial Fraction

$\Rightarrow y = \int \left(\frac{-1}{x} + \frac{1/2}{x+1} + \frac{1/2}{x-1} \right) \cdot dx$

$\Rightarrow y = -1 \cdot \log x + \frac{1}{2} \log(x+1) + \frac{1}{2} \log(x-1) + C$

$\Rightarrow y = \frac{1}{2} \left\{ -2 \log x + \log(x+1) + \log(x-1) \right\} + C$

$\Rightarrow y = \frac{1}{2} \left[\log \frac{(x+1)(x-1)}{x^2} \right] + C$

General Soln $\Rightarrow y = \frac{1}{2} \log \left(\frac{x^2-1}{x^2} \right) + C \rightarrow x=2, y=0$

Particular Soln.

$\Rightarrow y = \frac{1}{2} \log \left(\frac{x^2-1}{x^2} \right) - \frac{1}{2} \log \left(\frac{3}{4} \right)$

$0 = \frac{1}{2} \log \left(\frac{3}{4} \right) + C$

$C = -\frac{1}{2} \log \left(\frac{3}{4} \right)$

Q.13 $\cos\left(\frac{dy}{dx}\right) = a$; $(a \in \mathbb{R})$; $y=1$ when $x=0$

$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$

$\Rightarrow dy = \cos^{-1} a \cdot dx$ Integration

$\Rightarrow \int dy = \cos^{-1}(a) \cdot \int dx \Rightarrow \boxed{y = \cos^{-1} a \cdot (x) + c}$

$x=0, y=1$

$\Rightarrow 1 = \frac{\cos^{-1} a \cdot (0)}{0} + c \Rightarrow \boxed{c=1}$

Particular Solⁿ: $y = \cos^{-1} a \cdot x + 1$

$\Rightarrow y - 1 = \cos^{-1} a \cdot x$

$\Rightarrow \left(\frac{y-1}{x}\right) = \cos^{-1} a \Rightarrow \boxed{\cos\left(\frac{y-1}{x}\right) = a}$

Q.14 $\frac{dy}{dx} = y \tan x$; $y=1$ when $x=0$

$\Rightarrow \frac{dy}{y} = \tan x \cdot dx$ Integration

$\Rightarrow \int \frac{dy}{y} = \int \tan x \cdot dx \Rightarrow \boxed{\log(y) = \log \sec x + C}$

$x=0, y=1$ $(0, 1)$

$\log(1) = \log(\sec 0) + C \Rightarrow \boxed{C=0}$

Particular Solⁿ: $\log y = \log \sec x$

$\Rightarrow \boxed{y = \sec x}$

Q.15 Find the equation of a curve passing through the point (0,0) and whose differential equation is $y' = e^x \sin x$.

Ans. $\frac{dy}{dx} = e^x \sin x \Rightarrow dy = e^x \sin x dx$

⊗ Integration

$\Rightarrow \int dy = \int e^x \sin x dx$

Integration by Parts,

~~$y =$~~

ILATE

$$\int I \cdot II = I \int II - \int (I' \cdot II)$$

$\Rightarrow y = \sin x \cdot e^x - \int \cos x \cdot e^x dx$

ILATE

$\Rightarrow y = \sin x \cdot e^x - \left[\cos x \cdot e^x + \int \sin x \cdot e^x dx \right]$

$\Rightarrow y = \sin x \cdot e^x - \cos x \cdot e^x - y + C$

$\Rightarrow 2y = e^x (\sin x - \cos x) + C$

(0,0)
x y

$0 = e^0 (0 - 1) + C$

$\Rightarrow 0 = 1(-1) + C \Rightarrow 0 = -1 + C \Rightarrow \boxed{1 = C}$

$2y = e^x (\sin x - \cos x) + 1$

Exercise 9.4

Variable Separation method

Q.16 For the differential equation $xy \frac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through the point $(1, -1)$.

Ans. $xy \frac{dy}{dx} = (x+2)(y+2)$

C

$$\Rightarrow \frac{y}{y+2} dy = \frac{(x+2)}{x} dx \quad \text{Integration}$$

$$\Rightarrow \int \frac{y}{y+2} \cdot dy = \int \frac{x+2}{x} \cdot dx$$

$$\Rightarrow \int \frac{y+2-2}{y+2} \cdot dy = \int \left(1 + \frac{2}{x}\right) \cdot dx$$

$$\Rightarrow \int \left(1 - \frac{2}{y+2}\right) \cdot dy = x + 2 \log(x) + C$$

$$\Rightarrow \boxed{y - 2 \log(y+2) = x + 2 \log x + C} \quad \text{Curve}$$

$$-1 - 2 \log(1) = 1 + 2 \log 1 + C$$

Passes through $(1, -1)$

$$\underline{x=1, y=-1}$$

$$\boxed{-2 = C}$$

Again eqⁿ (Curve) $y - 2 \log(y+2) = x + 2 \log x - 2$

$$\Rightarrow y - x + 2 = \log(x^2) + \log(y+2)^2$$

$$\Rightarrow \boxed{y - x + 2 = \log(x^2 \cdot (y+2)^2)}$$

Q.17 Find the equation of a Curve passing through the point $(0, -2)$ given that at any point (x, y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

AOD

Slope of tangent = $\frac{dy}{dx}$

Ans. ATQ.

$\frac{dy}{dx} \times y = x$

$\Rightarrow y \cdot dy = x \cdot dx$
integration

$\Rightarrow \int y \cdot dy = \int x \cdot dx$

$\Rightarrow \boxed{\frac{y^2}{2} = \frac{x^2}{2} + C}$

Passes through $(0, -2)$

$\frac{(-2)^2}{2} = \frac{0}{2} + C$

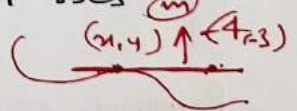
$\Rightarrow \boxed{2 = C}$

Curve $\left(\frac{y^2}{2} = \frac{x^2}{2} + 2 \right) \times 2$

$\Rightarrow y^2 = x^2 + 4$

$\Rightarrow \boxed{y^2 - x^2 = 4}$ ✓

Q.18 At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact $(-4, -3)$ to the point (x, y) . Find the equation of the curve given that it passes through $(-2, 1)$.



Ans. ATQ $\frac{dy}{dx} = 2 \left(\frac{y+3}{x+4} \right)$

$\Rightarrow \frac{dy}{y+3} = 2 \frac{dx}{x+4}$

$\Rightarrow \int \frac{dy}{y+3} = 2 \int \frac{dx}{x+4}$

$\log(y+3) = 2 \log(x+4) + C$

$(-2, 1)$

$\Rightarrow \log(4) = 2 \log(2) + C$

$\Rightarrow \log(4) = \log(4) + C$

$\boxed{0 = C}$

$$\Rightarrow \log(y+3) = 2 \log(x+4) + 0$$

$$\Rightarrow \log(y+3) = \log(x+4)^2$$

$$\Rightarrow \boxed{(y+3) = (x+4)^2} \quad \checkmark$$

2R1

Q.19 The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after 't' seconds.

Ans. $\left(\frac{\text{Rate of change in volume}}{V} \right) = \text{Constant } K$

$t=0$	$r=3$	\checkmark
$t=3$	$r=6$	\checkmark
$t=t$	$r=?$	

$$\Rightarrow \frac{dv}{dt} = K \quad \leftarrow \text{D.E.}$$

$$\Rightarrow dv = K \cdot dt$$

integration

$$\Rightarrow \int dv = K \int dt$$

$$\Rightarrow \boxed{V = Kt + C}$$

$$\Rightarrow \boxed{\frac{4}{3} \pi r^3 = Kt + C}$$

$t=0, r=3$ put

$$\Rightarrow \frac{4}{3} \pi (3^3) = K(0) + C$$

$$\Rightarrow \boxed{36\pi = C} \quad \checkmark$$

$$V = \frac{4}{3} \pi r^3$$

updated

$$\frac{4}{3} \pi r^3 = Kt + 36\pi$$

$t=3, r=6$ put

$$\Rightarrow \frac{4}{3} \pi 6^3 = K(3) + 36\pi$$

$$\Rightarrow 8 \times 36\pi = 3K + 36\pi$$

$$\Rightarrow 8 \times 36\pi - 36\pi = 3K$$

$$\Rightarrow 96\pi - 12\pi = 3K$$

$$\Rightarrow \boxed{K = 84\pi} \quad \checkmark$$

updated

$$\frac{4}{3} \pi r^3 = 84\pi t + 36\pi$$

Exercise 9.4

Q20, Q21, Q22, Q23

Q.20 In a bank, principal increases continuously at the rate of $\gamma\%$ per year. Find the value of γ if Rs 100 Double itself in 10 years. ($\log_e 2 = 0.6931$).

Ans. Rate of change of Principal = $\frac{dP}{dt} = \gamma$ (t = time years)

ATQ. $\frac{dP}{dt} = \gamma\% \text{ of } P$

$\Rightarrow \frac{dP}{dt} = \frac{\gamma}{100} \times P$ ← D.E.

$\Rightarrow \frac{dP}{P} = \left(\frac{\gamma}{100}\right) dt$

integration

$\Rightarrow \int \frac{dP}{P} = \frac{\gamma}{100} \int dt$

$\Rightarrow \log(P) = \frac{\gamma}{100} \cdot t + C$

Time line

t=0	P=100/—
t=10	P=200/—

$\log P = \frac{\gamma t}{100} + C$

t=0, P=100 put

$\Rightarrow \log 100 = C$

updated $\log P = \frac{\gamma t}{100} + \log 100$

2nd condⁿ. t=10, P=200 put

$\Rightarrow \log(200) = \frac{\gamma(10)}{100} + \log 100$

$\Rightarrow \log 200 - \log 100 = \frac{\gamma}{10}$

$\Rightarrow \log_e(2) = \frac{\gamma}{10}$

$\Rightarrow 10(0.6931) = \gamma \Rightarrow \gamma = 6.931\%$

$\log m - \log n = \log \frac{m}{n}$

put

Q.21 In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it worth after 10 years. ($e^{0.5} = 1.648$)

Ans. Principal = P

Rate of change of principal = $\frac{dP}{dt}$ $t \rightarrow$ time years

$$\frac{dP}{dt} = 5\% \text{ of } 'P'$$

$$\Rightarrow \frac{dP}{dt} = \frac{5}{100} \times P \} \rightarrow \underline{\underline{DE}}$$

<u>Time line</u>	
$t=0$	$P=1000$
$t=10$	$P=?$

$$\Rightarrow \frac{dP}{P} = \frac{1}{20} \cdot dt \quad \text{(integrate)}$$

$$\Rightarrow \int \frac{dP}{P} = \frac{1}{20} \int dt \Rightarrow$$

$$\Rightarrow \log(P) = \frac{1}{20}(t) + C$$

$$t=0, P=1000$$

$$\Rightarrow \log(1000) = C$$

$$\therefore \log P = \frac{t}{20} + \log(1000)$$

$$t=10 \quad P=?$$

$$\Rightarrow \log P = \frac{10}{20} + \log 1000$$

$$\Rightarrow \log P = \frac{1}{2} + \log 1000$$

\rightarrow exponential

$$\Rightarrow \log P = \frac{1}{2} + \log 1000$$

$$\Rightarrow P = e^{\left(\frac{1}{2} + \log 1000\right)}$$

$$a^{m+n} = a^m \cdot a^n$$

$$\Rightarrow P = e^{\frac{1}{2}} \cdot e^{\log 1000}$$

$$\Rightarrow P = (1.648) \times 1000$$

$$a^{\log_a b} = b$$

$$\Rightarrow P = \underline{\underline{₹1648}}$$

Q.22 In a culture, the bacteria count is 1,00,000.

The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Rate = $\frac{dy}{dx}$
 \propto

Time line	
(i) $t=0$	$B = 1,00,000$
(ii) $t=2$	$B = 1,00,000 + 10\% \text{ of } 1,00,000$ $= 1,00,000 + \frac{10}{100} \times 1,00,000$ $= 1,10,000$
$t = t \text{ hours}$?	$B = 2,00,000$

A.T.O.

Number of bacteria present = B

$\frac{dB}{dt} \propto B$

$\Rightarrow \frac{dB}{dt} = K \cdot B$ ← D.E.

Variable Separation

$\Rightarrow \frac{dB}{B} = K \cdot dt$

integration,

$\Rightarrow \int \frac{dB}{B} = K \int dt$

$\Rightarrow \log B = Kt + C$

$\log B = Kt + C$

(i) $t=0, B=1,00,000$

$\Rightarrow \log 1,00,000 = C$

$\log B = Kt + \log(1,00,000)$

(ii) $t=2, B=1,10,000$

$\log(1,10,000) = K(2) + \log 1,00,000$

$\Rightarrow \log \left(\frac{1,10,000}{1,00,000} \right) = 2K$

$\Rightarrow K = \frac{1}{2} \log \left(\frac{11}{10} \right)$

$$\log B = k \cdot t + \log(100000)$$

$$k = \frac{1}{2} \log\left(\frac{11}{10}\right)$$

$$\log B = \frac{1}{2} \log\left(\frac{11}{10}\right) \cdot t + \log(100000)$$

$$t = ? , B = 200000$$

$$\Rightarrow \log(200000) = \frac{1}{2} \log\left(\frac{11}{10}\right) \cdot t + \log(100000)$$

$$\Rightarrow \log\left(\frac{200000}{100000}\right) = \frac{1}{2} \log\left(\frac{11}{10}\right) \times t?$$

$$\Rightarrow \frac{2 \log 2}{\log\left(\frac{11}{10}\right)} = t$$

hours

Q. 23 The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is —

(A) $e^x + e^{-y} = c$ (B) $e^x + e^y = c$
 (C) $e^{-x} + e^y = c$ (D) $e^{-x} + e^{-y} = c$

$$\frac{dy}{dx} = e^x \cdot (e^y)$$

$$\Rightarrow \frac{dy}{e^y} = e^x \cdot dx$$

$$\Rightarrow e^{-y} \cdot dy = e^x \cdot dx$$

integration

$$\Rightarrow \int e^{-y} \cdot dy = \int e^x \cdot dx$$

$$\Rightarrow \frac{e^{-y}}{-1} = e^x + c$$

$$\Rightarrow -e^{-y} = e^x + c$$

$$\Rightarrow -e^{-y} - e^x = c$$

$$\Rightarrow e^{-y} + e^x = -c$$

$$-c = c_1$$

$$\Rightarrow e^{-y} + e^x = c_1$$

Homogeneous Differential Equations : \rightarrow

[समघातीय अवकल समीकरण] \rightarrow what is it ?
 \rightarrow How to solve ?

Homogeneous Functions

$f(x, y) \rightarrow$ Homogeneous Funcⁿ.
of Degree n

if $f(\lambda x, \lambda y) = \lambda^n \cdot f(x, y)$

e.g. $f_1(x, y) = y^2 + 2xy$

$f_1(\lambda x, \lambda y) = (\lambda y)^2 + 2(\lambda x)(\lambda y)$
 $= \lambda^2 y^2 + 2\lambda^2 xy$
 $= \lambda^2 (y^2 + 2xy) = \lambda^2 \cdot f_1(x, y)$

e.g. $f_2(x, y) = 2x + 3y + 5$

$f_2(\lambda x, \lambda y) = 2\lambda x + 3\lambda y + 5$
 $\neq \lambda^n \cdot f_2(x, y)$

e.g. $f_3(x, y) = \cos\left(\frac{y}{x}\right)$

$f_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right)$
 $= \lambda^0 \cdot \cos\left(\frac{y}{x}\right)$

e.g. $f_4(x, y) = \sin x + \sin y$

$f_4(\lambda x, \lambda y) = \sin \lambda x + \sin \lambda y$
 $\neq \lambda^n \cdot f_4(x, y)$

*

Homogeneous Differential Eqⁿ.

$\frac{dy}{dx} = f(x, y) \rightarrow$ Homo. Diff. Eqⁿ.

if $f(\lambda x, \lambda y) = \lambda^0 \cdot f(x, y)$

Homo. function of degree 0.

e.g. $(x-y) \frac{dy}{dx} = (x+2y)$

$\Rightarrow \frac{dy}{dx} = \frac{x+2y}{x-y} = f(x, y)$

$f(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \lambda^0 \left(\frac{x+2y}{x-y}\right)$

e.g. $(1 + e^{\frac{x}{y}}) dx = e^{\frac{x}{y}} \left(\frac{x}{y} - 1\right) dy$

$\Rightarrow \frac{dx}{dy} = \frac{e^{\frac{x}{y}} \left(\frac{x}{y} - 1\right)}{1 + e^{\frac{x}{y}}} = f(x, y)$

$f(\lambda x, \lambda y) = \frac{e^{\frac{\lambda x}{\lambda y}} \left(\frac{\lambda x}{\lambda y} - 1\right)}{1 + e^{\frac{\lambda x}{\lambda y}}}$

$= \lambda^0 \cdot f(x, y)$

How to Solve Homogeneous Differential Equation?

Case-I कुछ Homo. Diff. Eqⁿ में बहुत सारी जगह $\frac{dy}{dx}$ बन जाता है i.e. $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, तो इनमें हम $\frac{y}{x} = v$

मतलब $y = vx$ substitute कर देते हैं Solve -

Case-II. कुछ Homo. Diff. Eqⁿ में $\frac{x}{y}$ देखने को मिलता है, i.e. $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$, तो इनमें हम $\frac{x}{y} = v$ मतलब

$x = v \cdot y$ substitute कर देते हैं, Solve -

e.g. Show that the differential equation

$x \cos\left(\frac{y}{x}\right) \left(\frac{dy}{dx}\right) = y \cos\left(\frac{y}{x}\right) + x$ is Homogeneous & Solve it.

Ans. $\frac{dy}{dx} = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)} = f(x, y)$

$$f(\lambda x, \lambda y) = \frac{\cancel{\lambda} y \cdot \cos\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right) + \cancel{\lambda} x}{\cancel{\lambda} x \cdot \cos\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right)} = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)}$$

∴ Homogeneous Diff. Eqⁿ

$$\left(\frac{dy}{dx}\right) = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cdot \cos\left(\frac{y}{x}\right)} = \frac{\left(\frac{y}{x}\right) \cdot \cos\left(\frac{y}{x}\right) + 1}{\cos\left(\frac{y}{x}\right)} \quad \text{--- (1)}$$

Let $\frac{y}{x} = v$ $\Rightarrow y = vx$ (Substitution)

$$\frac{dy}{dx} = \frac{d(vx)}{dx} = v \cdot 1 + x \cdot \frac{dv}{dx}$$

Subs. $\Rightarrow v + x \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v = \frac{v \cos v + 1 - v \cos v}{\cos v}$$

Variable Separation ✓

$$\Rightarrow \cos v \cdot dv = \frac{dx}{x}$$

$$\Rightarrow \int \cos v \cdot dv = \int \frac{dx}{x} \quad \text{(integration)}$$

$$\Rightarrow \sin v = \log|x| + C$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \log|x| + C$$

e.g. Show that the differential eqⁿ

$2y e^{\frac{x}{y}} \cdot dx + (y - 2x e^{\frac{x}{y}}) \cdot dy = 0$ is Homogeneous and find its particular solution, given that $x=0$ & $y=1$.

Ans. $2y e^{\frac{x}{y}} \cdot (dx) = (2x e^{\frac{x}{y}} - y) (dy)$

$$\Rightarrow \frac{dx}{dy} = \frac{2x e^{\frac{x}{y}} - y}{2y e^{\frac{x}{y}}} = \frac{2x e^{\frac{x}{y}}}{2y e^{\frac{x}{y}}} - \frac{y}{2y e^{\frac{x}{y}}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} - \frac{1}{2e^{\frac{x}{y}}}$$

$$f(x, y) = \frac{x}{y} - \frac{1}{2e^{\frac{x}{y}}} = \int \left(\frac{x}{y} - \frac{1}{2e^{\frac{x}{y}}} \right)$$

Homo. Diff. eqⁿ

$$\frac{dx}{dy} = \frac{x}{y} - \frac{1}{2e^{\frac{x}{y}}}$$

Substitution:

$$\frac{x}{y} = v \Rightarrow x = vy$$

$$\Rightarrow \frac{dx}{dy} = \frac{d(vy)}{dy} = v + y \cdot \frac{dv}{dy}$$

$$\Rightarrow v + y \cdot \frac{dv}{dy} = v - \frac{1}{2e^v}$$

$$\Rightarrow 2e^v \cdot dv = -\frac{dy}{y}$$

Integration.

$$\Rightarrow 2 \int e^v \cdot dv = - \int \frac{dy}{y}$$

$$\Rightarrow 2 \int e^v \cdot dv = - \int \frac{dy}{y}$$

$$\Rightarrow \boxed{2(e^v) = -\log(y) + c}$$

$$v = \frac{x}{y}$$

$$\Rightarrow \boxed{2e^{x/y} = -\log(y) + c} \quad \star \text{ General sol}^n$$

$$(x=0, y=1) \text{ put}$$

$$\Rightarrow 2e^0 = -\log(1) + c$$

$$\Rightarrow 2 \times 1 = -0 + c \quad \Rightarrow \boxed{c=2}$$

Particular Solⁿ

$$\boxed{2e^{x/y} = -\log(y) + 2}$$

Exercise 9.5

 (Homogeneous Differential Equations)

Show that the given differential equations are homogeneous and solve each of them.

Q.1 $(x^2 + xy).dy = (x^2 + y^2).dx$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} = f(x, y)}$$

divide by x^2 in both Nr. & Dr.

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2 + y^2)/x^2}{(x^2 + xy)/x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)}$$

Subs.

$$\Rightarrow v + x \frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v^2}{1 + v} - \frac{v}{1} \Rightarrow x \frac{dv}{dx} = \frac{1 + v^2 - v - v^2}{1 + v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 - v}{1 + v} \Rightarrow x \frac{dv}{dx} = - \left(\frac{v - 1}{v + 1} \right)$$

$$\Rightarrow \left(\frac{v + 1}{v - 1} \right) dv = - \frac{dx}{x}$$

integration

$$f(x, y) = \frac{(dx)^2 + (dy)^2}{(dx)^2 + (dx) \cdot (dy)}$$

$$= \frac{x^2(x^2 + y^2)}{x^2(x^2 + xy)}$$

$$= f^0(x, y)$$

Homogeneous Diff. eqⁿ.

Substitution.

$$\boxed{\frac{y}{x} = v}$$

$$\Rightarrow y = xv$$

$$\boxed{\frac{dy}{dx} = \frac{d(xv)}{dx} = v + x \frac{dv}{dx}}$$

$$\Rightarrow \int \left(\frac{v+1}{v-1} \right) \cdot dv = - \int \frac{dn}{x}$$

$$\sqrt{x}$$

$$\Rightarrow \int \frac{(v-1)+2}{(v-1)} \cdot dv = - \int \frac{dn}{x}$$

$$\Rightarrow \int \left(1 + \frac{2}{v-1} \right) \cdot dv = - \log x + C$$

$$\Rightarrow \left[v + 2 \log(v-1) \right] = - \log x + C$$

$$v = \frac{y}{x}$$

$$\Rightarrow \frac{y}{x} + 2 \log \left(\frac{y}{x} - 1 \right) + \log x = C$$

$$\Rightarrow \log \left(\frac{y-x}{x} \right)^2 + \log x = - \frac{y}{x} + C$$

$$\Rightarrow \log \left(\frac{(y-x)^2}{x^2} \cdot x \right) = \left(- \frac{y}{x} + C \right)$$

$$\Rightarrow \frac{(y-x)^2}{x} = e^{-\frac{y}{x} + C}$$

$$a^{m+n} = a^m \cdot a^n$$

$$\Rightarrow (y-x)^2 = x \cdot e^{-\frac{y}{x}} \cdot e^C$$

$$\Rightarrow (y-x)^2 = x e^{-\frac{y}{x}} \cdot C_1$$

$e^C \rightarrow \text{const}$
 $e^C \rightarrow \text{const}$
 $\rightarrow \text{New Const.} = C_1$

Q.2 $y' = \frac{x+y}{x}$

$\Rightarrow \frac{dy}{dx} = 1 + \frac{y}{x} = f(x,y)$

$\Rightarrow \left(\frac{dy}{dx}\right) = 1 + \left(\frac{y}{x}\right)$

Substitution,

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \frac{dy}{dx} = \left(v + x \frac{dv}{dx}\right)$

Subs.

$\Rightarrow x + x \frac{dv}{dx} = 1 + x$

$\Rightarrow dv = \frac{dx}{x} \Rightarrow \int dv = \int \frac{dx}{x} \Rightarrow v = \log|x| + C$

$\Rightarrow \frac{y}{x} = \log|x| + C \Rightarrow y = x \log|x| + Cx$

$f(\lambda x, \lambda y) = 1 + \frac{\lambda y}{\lambda x}$
 $= 1 + \frac{y}{x}$
 $= \lambda^0 \cdot f(x,y)$
 Homo. Diff. eqn.

Q.3 $(x-y) dy - (x+y) dx = 0$

$\Rightarrow (x-y) \cdot dy = (x+y) dx$

$\Rightarrow \frac{dy}{dx} = \left(\frac{x+y}{x-y}\right) = f(x,y)$

$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{x(1 + \frac{y}{x})}{x(1 - \frac{y}{x})}$

Substitution

$\frac{y}{x} = v \Rightarrow y = vx$

$\frac{dy}{dx} = v + x \frac{dv}{dx}$

$f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y}$
 $= \lambda^0 \left(\frac{x+y}{x-y}\right)$
 $= \lambda^0 (f(x,y))$
 Homo. Diff. eqn.

$$\Rightarrow \left(v + x \frac{dv}{dx} \right) = \frac{1+v}{1-v} \quad (\text{after substitution})$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v - v + v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \underbrace{V.S.}_{(1-v) dv} = \frac{dx}{x} \quad (\text{integration})$$

$$\Rightarrow \int \frac{1-v}{1+v^2} \cdot dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{1+v^2} dv - \frac{1}{2} \int \frac{2v}{1+v^2} dv = \log|x| + C$$

$$\Rightarrow \tan^{-1}(v) - \frac{1}{2} \log|1+v^2| = \log|x| + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \log \left| 1 + \frac{y^2}{x^2} \right| + \log|x| + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \left\{ \log \left| \frac{x^2+y^2}{x^2} \right| + 2 \log|x| \right\} + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \left\{ \log \left| \frac{x^2+y^2}{x^2} \times x^2 \right| \right\} + C$$

$$\Rightarrow \boxed{\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \log(x^2+y^2) + C}$$

$$\boxed{Q.4} \quad (x^2 - y^2) dx + 2xy dy = 0$$

$$\Rightarrow 2xy \cdot dy = (y^2 - x^2) \cdot dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = f(x, y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 \left(\frac{y^2}{x^2} - 1 \right)}{x^2 \left(2 \frac{y}{x} \right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{y}{x} \right)^2 - 1}{2 \left(\frac{y}{x} \right)}$$

↓ ↓ ↓

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - \frac{v}{1} = \frac{v^2 - 1 - 2v^2}{2v} = \frac{-v^2 - 1}{2v} = \frac{-(v^2 + 1)}{2v}$$

$$\Rightarrow \left(\frac{2v}{v^2 + 1} \right) \cdot dv = - \frac{dx}{x}$$

integration

$$\Rightarrow \int \frac{2v}{v^2 + 1} dv = - \int \frac{dx}{x}$$

$$\Rightarrow \log(v^2 + 1) = - \log x + \log C$$

$$\Rightarrow \log \left(\frac{y^2}{x^2} + 1 \right) = \log \left(\frac{C}{x} \right)$$

$$f(x, y) = \frac{(xy)^2 - (yx)^2}{2(xy)(xy)}$$

$$= \frac{x^2(y^2 - x^2)}{x^2(2xy)}$$

$$= \int^0 (f(x, y))$$

H. O. E

Substitution

$$\frac{y}{x} = v \Rightarrow y = xv$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\frac{y^2 + x^2}{x^2} = \frac{C}{x}$$

$$\Rightarrow y^2 + x^2 = Cx$$

Q.5 $x^2 \left(\frac{dy}{dx} \right) = x^2 - 2y^2 + xy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = 1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \rightarrow f(x,y)$$

$$\begin{aligned} f(x,y) &= 1 + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) \\ &= \int^0 (f(x,y)) \end{aligned}$$

H.D.E ✓

Substitution:

$$\frac{y}{x} = v$$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = 1 - 2v^2 + v$$

$$\Rightarrow \frac{dv}{1 - 2v^2} = \frac{dx}{x}$$

integration

$$\Rightarrow \int \frac{dv}{1 - (\sqrt{2}v)^2} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2 \times 1} \cdot \log \left| \frac{1 + \sqrt{2}v}{1 - \sqrt{2}v} \right| = \log|x| + c$$

$$v = \frac{y}{x}$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log \left| \frac{1 + \sqrt{2} \cdot \frac{y}{x}}{1 - \sqrt{2} \cdot \frac{y}{x}} \right| = \log|x| + c$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log \left| \frac{x + \sqrt{2}y}{x - \sqrt{2}y} \right| = \log|x| + c$$

Exercise 9.5

(Homogeneous Diff. Eqⁿ.)

Show that the given diff. eqⁿ are Homogeneous & solve each of them.

$$\boxed{\text{Q.6}} \quad \frac{x dy - y dx}{dx} = \sqrt{x^2 + y^2} \cdot \frac{dx}{dx}$$

$$\Rightarrow x \left(\frac{dy}{dx} \right) - y = \sqrt{x^2 + y^2}$$

$$\Rightarrow x \cdot \frac{dy}{dx} = \sqrt{x^2 + y^2} + y$$

$$\Rightarrow \left[\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x} \right] = f(x, y)$$

$$\Rightarrow \left(\frac{dy}{dx} \right) = x \left\{ \sqrt{1 + \left(\frac{y}{x} \right)^2} + \left(\frac{y}{x} \right) \right\}$$

Substitution:

$$\frac{y}{x} = v \Rightarrow \boxed{y = xv}$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow \cancel{x} + x \frac{dv}{dx} = \sqrt{1 + v^2} + \cancel{x}$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integration

$$\Rightarrow \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} \Rightarrow \boxed{\log(v + \sqrt{1 + v^2}) = \log x + \log c}$$

$f(x, y)$

$$= \frac{\sqrt{(x)^2 + (y)^2} + y}{dx}$$

$$= \frac{x(\sqrt{1 + \left(\frac{y}{x}\right)^2} + \left(\frac{y}{x}\right))}{x}$$

x

$$= \int (f(x, y))$$

Homogeneous D.E.

$$\Rightarrow \log(v + \sqrt{1+v^2}) = \log x + \log c$$

Put $v = \frac{y}{x}$

$$\Rightarrow \log\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = \log(xc)$$

$$\Rightarrow \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}} = xc \Rightarrow \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = xc$$

$$\Rightarrow y + \sqrt{x^2+y^2} = x^2 \cdot c$$

Q.7 $\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y \cdot dx = \left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x \cdot dy$

$$\Rightarrow \frac{\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} \cdot y}{\left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} \cdot x} = \frac{dy}{dx}$$

$$\Rightarrow \frac{\frac{y}{x} \left\{ \cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right) \cdot \sin\left(\frac{y}{x}\right) \right\}}{\left\{ \frac{y}{x} \cdot \sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right) \right\}} = \frac{dy}{dx}$$

$f(x, y)$

H. D. E.

$$f(\lambda x, \lambda y) = \lambda^0 \cdot f(x, y)$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} \cdot \left\{ \cos \frac{y}{x} + \frac{y}{x} \cdot \sin \frac{y}{x} \right\}}{\frac{y}{x} \cdot \sin \frac{y}{x} - \cos \frac{y}{x}}$$

Substitution,

$$\frac{y}{x} = v \Rightarrow y = v \cdot x$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v \cdot (\cos v + v \sin v)}{v \sin v - \cos v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + v^2 \sin v}{v \sin v - \cos v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + \cancel{v^2 \sin v} - v^2 \sin v + v \cos v}{v \sin v - \cos v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v \cos v}{v \sin v - \cos v} \quad (\text{Variable separation})$$

$$\Rightarrow \left(\frac{v \sin v - \cos v}{v \cos v} \right) \cdot dv = 2 \cdot \frac{dx}{x} \quad (\text{Integrate})$$

$$\Rightarrow \int \left(\tan v - \frac{1}{v} \right) \cdot dv = 2 \int \frac{dx}{x}$$

$$\Rightarrow \log |\sec v| - \log |v| = 2 \log |x| + \log(C)$$

$$\Rightarrow \log \left| \frac{\sec v}{v} \right| = \log |x^2 \cdot C|$$

$$v = \frac{y}{x}$$

$$\Rightarrow \frac{\sec v}{v} = x^2 \cdot c$$

$$\Rightarrow \frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)} = x^2 \cdot c$$

$$\Rightarrow \boxed{\sec\left(\frac{y}{x}\right) = \frac{xy \cdot c}{x}}$$

$$v = \frac{y}{x}$$

$$\left(\frac{1}{c}\right) = xy \cdot \cos\left(\frac{y}{x}\right)$$

$$\boxed{C_1 = xy \cdot \cos\left(\frac{y}{x}\right)}$$

[Q.8] $x \left(\frac{dy}{dx}\right) - y + x \sin\left(\frac{y}{x}\right) = 0$

$$\Rightarrow x \cdot \frac{dy}{dx} = y - x \sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right)}$$

$$f(x, y)$$

$$= \frac{dy}{dx} - \sin\left(\frac{y}{x}\right)$$

$$= \frac{y}{x} - \sin\left(\frac{y}{x}\right)$$

$$= \int f(x, y)$$

H. D. E

Substitution: \rightarrow

$$\frac{y}{x} = v \Rightarrow y = xv$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \frac{dv}{\sin v} = - \frac{dx}{x} \quad \text{integration}$$

$$\Rightarrow \int \operatorname{cosec} v \cdot dv = - \int \frac{dx}{x}$$

$$\Rightarrow \log | \csc v - \cot v | = - \log x + \log c$$

$$\Rightarrow \log \left| \frac{1}{\sin v} - \frac{\cos v}{\sin v} \right| = \log \left| \frac{c}{x} \right|$$

$$\Rightarrow \frac{(1 - \cos v)}{(\sin v)} = \frac{c}{x} \quad \left(v = \frac{y}{x} \right)$$

$$\Rightarrow x \left(1 - \cos \frac{y}{x} \right) = c \cdot \sin \left(\frac{y}{x} \right)$$

Q.9 $y \frac{dx}{dy} + x \log \left(\frac{y}{x} \right) \cdot \frac{dy}{dy} - 2x \frac{dy}{dy} = 0$

$$\Rightarrow y \cdot \left(\frac{dx}{dy} \right) + x \log \left(\frac{x}{y} \right)^{-1} - 2x = 0$$

$$\Rightarrow y \cdot \left(\frac{dx}{dy} \right) - x \log \left(\frac{x}{y} \right) - 2x = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{x \cdot \log \left(\frac{x}{y} \right) + 2x}{y}$$

$$\Rightarrow \left(\frac{dx}{dy} \right) = \frac{x}{y} \cdot \log \left(\frac{x}{y} \right) + 2 \left(\frac{x}{y} \right)$$

$\rightarrow f(x, y)$

$f(x, y)$
 $= x \log \left(\frac{x}{y} \right) + 2x$
H.D.F

Substitution

$$\left(\frac{x}{y} = v \right) \Rightarrow x = yv$$

$$\Rightarrow \frac{dx}{dy} = v + y \cdot \frac{dv}{dy}$$

$$\Rightarrow v + y \cdot \frac{dv}{dy} = v \cdot \log(v) + 2v$$

$$\Rightarrow \underline{y \cdot \frac{dv}{dy}} = \underline{v \cdot \log V + V} = \underline{v (\log V + 1)}$$

$$\Rightarrow \frac{dv}{v \cdot (\log V + 1)} = \frac{dy}{y}$$

(Integration)

$$\Rightarrow \int \frac{dv}{v \cdot (\log V + 1)} = \int \frac{dy}{y}$$

Legend

$$\Rightarrow \log |\log V + 1| = \log |y| + \log C$$

$$\Rightarrow \cancel{\log} |\log V + 1| = \cancel{\log} (y C) \quad \left(V = \frac{x}{y} \right)$$

$$\Rightarrow \boxed{\log \left(\frac{x}{y} \right) + 1 = y C}$$

Q.10 $(1 + e^{\frac{x}{y}}) \cdot \underline{dx} + e^{\frac{x}{y}} \left(1 - \frac{x}{y} \right) \cdot \underline{dy} = 0$

$$\Rightarrow (1 + e^{\frac{x}{y}}) \cdot (dx) = e^{\frac{x}{y}} \cdot \left(\frac{x}{y} - 1 \right) \cdot dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^{\frac{x}{y}} \cdot \left(\frac{x}{y} - 1 \right)}{(1 + e^{\frac{x}{y}})} = f(x, y)$$

$$f(\lambda x, \lambda y) = \lambda^0 \cdot f(x, y) \quad \text{H.D.E}$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^{\frac{x}{y}} \left(\frac{x}{y} - 1 \right)}{1 + e^{\frac{x}{y}}}$$

↓
After subst.

Subst.

$$\frac{x}{y} = v$$

$$\Rightarrow x = y \cdot v$$

$$\Rightarrow \frac{dx}{dy} = v + y \cdot \frac{dv}{dy}$$

$$\Rightarrow v + y \cdot \frac{dv}{dy} = \frac{e^v (v-1)}{1+e^v}$$

$$\Rightarrow y \cdot \frac{dv}{dy} = \frac{e^v \cdot v - e^v}{1+e^v} - v = \frac{v \cdot e^v - e^v - v - v \cdot e^v}{1+e^v}$$

$$\Rightarrow y \cdot \frac{dv}{dy} = - \frac{(e^v + v)}{(e^v + 1)} \Rightarrow \left(\frac{e^v + 1}{e^v + v} \right) dv = - \frac{dy}{y}$$

Integration.

$$\Rightarrow \int \frac{e^v + 1}{e^v + v} dv = - \int \frac{dy}{y}$$

$$v = \frac{x}{y}$$

$$\Rightarrow \log(e^v + v) = - \log y + \log c$$

$$\Rightarrow \log \left(e^{\frac{x}{y}} + \frac{x}{y} \right) = \log \left(\frac{c}{y} \right)$$

$$\Rightarrow \left(e^{\frac{x}{y}} + \frac{x}{y} \right) = \frac{c}{y}$$

$$\Rightarrow y \cdot e^{\frac{x}{y}} + x = c$$

Exercise 9.5 (Homogeneous Differential Equations)

Find the particular solution satisfying the given condition:

Q.11 $(x+y).dy + (x-y).dx = 0$; $y=1$ when $x=1$.

$$\Rightarrow (x+y)dy = (y-x).dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x} = \frac{x(\frac{y}{x}-1)}{x(\frac{y}{x}+1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1}$$

Substitution,

$$\frac{y}{x} = v$$

$$\Rightarrow y = xv$$

$$\Rightarrow \frac{dy}{dx} = \frac{d(xv)}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v-1}{v+1} - \frac{v}{1} = \frac{v-1-v^2-v}{v+1} = -\frac{(v^2+1)}{(v+1)}$$

Variable separation

$$\Rightarrow \left(\frac{v+1}{v^2+1} \right) dv = -\frac{dx}{x} \quad \text{integration,}$$

$$\Rightarrow \int \left(\frac{v}{v^2+1} + \frac{1}{v^2+1} \right) \cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \int \frac{2v}{v^2+1} \cdot dv + \int \frac{1}{v^2+1} \cdot dv = -\log|x| + C$$

$$\Rightarrow \frac{1}{2} \log|v^2+1| + \tan^{-1}v = -\log|x| + C$$

$$v = \frac{y}{x}$$

$$\Rightarrow \left(\frac{1}{2} \log(y^2+1) + \tan^{-1} y = -\log|x| + C \right) \times 2$$

$$v = \frac{y}{x}$$

$$\Rightarrow \log\left(\frac{y^2}{x^2} + 1\right) + 2 \tan^{-1}\left(\frac{y}{x}\right) = \underline{\underline{2 \log|x| + 2C}}$$

$$\Rightarrow \log\left(\frac{y^2+x^2}{x^2}\right) + \log(x^2) + 2 \tan^{-1}\left(\frac{y}{x}\right) = 2C$$

$$\Rightarrow \log\left(\frac{y^2+x^2}{x^2} \times x^2\right) + 2 \tan^{-1}\left(\frac{y}{x}\right) = 2C$$

$$\Rightarrow \boxed{\log(y^2+x^2) + 2 \tan^{-1}\left(\frac{y}{x}\right) = \underline{\underline{2C}}}$$

$$y=1, x=1 \text{ put}$$

$$\Rightarrow \log(2) + 2 \tan^{-1}(1) = 2C$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

$$\Rightarrow \log 2 + 2 \times \frac{\pi}{4} = 2C$$

$$\Rightarrow \log 2 + \frac{\pi}{2} = \underline{\underline{2C}}$$

Particular Solution:

$$\log(y^2+x^2) + 2 \tan^{-1}\left(\frac{y}{x}\right) = \log 2 + \frac{\pi}{2}$$

$$\boxed{\text{Q.12}} \quad x^2 dy + (xy + y^2) dx = 0 ; y=1 \text{ when } x=1$$

$$\Rightarrow x^2 dy = -(xy + y^2) dx$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{xy + y^2}{x^2}\right)$$

$$\text{Substitution: } \frac{y}{x} = v$$

$$\Rightarrow y = xv$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\left[\frac{y}{x} + \left(\frac{y}{x}\right)^2\right]$$

Substitution:

$$v + x \frac{dv}{dx} = -(v + v^2)$$

$$\Rightarrow x \frac{dv}{dx} = -(2v + v^2)$$

V.S.

$$\Rightarrow \frac{dv}{2v + v^2} = - \frac{dx}{x}$$

integration

$$\Rightarrow \int \frac{dv}{v(v+2)} = - \int \frac{dx}{x}$$

$$\frac{A}{v} + \frac{B}{v+2}$$

By partial fraction

$$\Rightarrow \int \left(\frac{\frac{1}{2}}{v} - \frac{\frac{1}{2}}{v+2} \right) \cdot dv = - \log|x| + \log c$$

$$\Rightarrow \frac{1}{2} \log|v| - \frac{1}{2} \log|v+2| = - \log|x| + \log c$$

$$\Rightarrow \frac{1}{2} \left\{ \log \left| \frac{v}{v+2} \right| \right\} = \log \left| \frac{c}{x} \right|$$

$$\Rightarrow \log \left| \frac{v}{v+2} \right| = 2 \log \left| \frac{c}{x} \right|$$

$v = \frac{y}{x}$
 $n \cdot \log m = \log m^n$

$$\Rightarrow \log \left| \frac{\frac{y}{x}}{\frac{y}{x} + 2} \right| = \log \left(\frac{c^2}{x^2} \right)$$

$$\Rightarrow \frac{\left(\frac{y}{x} \right)}{\left(\frac{y+2x}{x} \right)} = \frac{c^2}{x^2} \Rightarrow \frac{y}{y+2x} = \frac{c^2}{x^2}$$

$$\Rightarrow \boxed{y x^2 = (y+2x) \cdot c^2} \quad \text{with } \boxed{x=1, y=1}$$

$$\Rightarrow 1 = 3 \cdot c^2$$

$$\Rightarrow c^2 = \frac{1}{3}$$

$$y x^2 = (y+2x) \cdot \frac{1}{3}$$

$$\Rightarrow \boxed{3y x^2 = (y+2x)}$$

Q.13 $\left[x \sin^2\left(\frac{y}{x}\right) - y \right] dx + x dy = 0$; $y = \frac{\pi}{4}$ when $x=1$

$\Rightarrow x \cdot dy = \left(y - x \cdot \sin^2\left(\frac{y}{x}\right) \right) \cdot dx$ | Substitution.

$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x} \right) - \sin^2\left(\frac{y}{x}\right)$

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$\Rightarrow \cancel{x} + x \frac{dv}{dx} = \cancel{x} - \sin^2(v)$

$\Rightarrow \frac{dv}{\sin^2 v} = - \frac{dx}{x}$ (integration)

$\Rightarrow \int \frac{dv}{\sin^2 v} = - \int \frac{dx}{x} \Rightarrow \int \csc^2 v \cdot dv = - \int \frac{dx}{x}$

$\Rightarrow -\cot v = -\log|x| + C$

$v = \frac{y}{x}$

$\Rightarrow -\cot\left(\frac{y}{x}\right) = -\log|x| + C$

$x=1, y = \frac{\pi}{4}$

$-\cot\left(\frac{\pi}{4}\right) = -\log|1| + C$

$\Rightarrow \cancel{-}\cot\left(\frac{y}{x}\right) = \cancel{-}\log|x| + 1$

$\Rightarrow -1 = C$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log_e|x| + 1$

$1 = \log_e e$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log_e|x| + \log_e e$

$\log m + \log n = \log mn$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log_e|x|$

Q.14 $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$; $y=0$ when $x=1$.

$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

Substitution.

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \cancel{y} + x \frac{dv}{dx} = \cancel{y} - \operatorname{cosec}(v)$

$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$\Rightarrow \frac{dv}{\operatorname{cosec} v} = - \frac{dx}{x}$

integration

$\Rightarrow \int \sin v \cdot dv = - \int \frac{dx}{x} \Rightarrow -\cos v = -\log|x| - c$

$\Rightarrow \cancel{\cos v} \quad \boxed{\cos v = \log|x| + c} \quad \left(v = \frac{y}{x}\right)$

$\Rightarrow \boxed{\cos\left(\frac{y}{x}\right) = \log|x| + c}$

$y=0, x=1$

$\cos 0 = \log|1| + c$

$\Rightarrow \boxed{1 = c}$

$\Rightarrow \boxed{\cos\left(\frac{y}{x}\right) = \log|x| + 1}$

$\Rightarrow \cos \frac{y}{x} = \log|x| + \log e$

$\Rightarrow \boxed{\cos \frac{y}{x} = \log|xe|}$

Q.15 $2xy + y^2 - 2x^2 \left(\frac{dy}{dx}\right) = 0$; $y=2$ when $x=1$

$\Rightarrow 2xy + y^2 = 2x^2 \left(\frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{2xy}{2x^2} + \frac{y^2}{2x^2} = \frac{y}{x} + \frac{1}{2} \left(\frac{y}{x}\right)^2$

$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) + \frac{1}{2} \left(\frac{y}{x}\right)^2$

$f\left(\frac{y}{x}\right)$

$\Rightarrow \cancel{x} + x \frac{dv}{dx} = \cancel{x} + \frac{1}{2} v^2$

$\Rightarrow \frac{dv}{v^2} = \frac{1}{2} \frac{dx}{x}$ Integrate

$\Rightarrow \int \frac{dv}{v^2} = \frac{1}{2} \int \frac{dx}{x} \Rightarrow \int v^{-2} \cdot dv = \frac{1}{2} \log|x| + C$

$\Rightarrow \frac{1}{v} = \frac{1}{2} \log|x| + C \Rightarrow \boxed{-\frac{x}{y} = \frac{1}{2} \log|x| + C}$

$\Rightarrow -\frac{x}{y} = \frac{1}{2} (\log|x|) - \frac{1}{2}$

$\Rightarrow -\frac{x}{y} = \frac{\log|x| - 1}{2}$

$\Rightarrow \frac{-2x}{\log|x| - 1} = y$

$\Rightarrow y = \frac{2x}{1 - \log|x|}$

Substitution,

$\frac{y}{x} = v \Rightarrow y = xv$

$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$x=1, y=2$

$\rightarrow -\frac{1}{2} = \frac{1}{2} \log|1| + C$

$\Rightarrow C = -\frac{1}{2}$

Q.16 & Q.17
Solved in Video
Directly.

Linear Differential Equations [रैखिक अवकल समीकरण]

Pattern-1 ★

$$\frac{dy}{dx} + p \cdot y = Q$$

↓
↓

Constant or functions of x

e.g. $\frac{dy}{dx} + 1 \cdot y = \sin x$

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$$

Pattern-2

$$\frac{dx}{dy} + p \cdot x = Q$$

↓
↓

Constant or functions of y

e.g. $\frac{dx}{dy} - \frac{x}{y} = 2y$

$$\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y$$

Steps to Solve $\frac{dy}{dx} + p \cdot y = Q$

(i) Find the Integrating Factor

I.F. = $e^{\int p \cdot dx}$

↑
(समाकलन गुणक)

(ii) Solution:

$y \cdot (\text{I.F.}) = \int (Q \times \text{I.F.}) \cdot dx + c$

Steps to Solve $\frac{dx}{dy} + p \cdot x = Q$

(i) Find the Integrating Factor

I.F. = $e^{\int p \cdot dy}$

(ii) Solution

$x \cdot (\text{I.F.}) = \int Q(\text{I.F.}) \cdot dy + c$

e.g. Find the particular solution of the differential

equation $\left(\frac{dy}{dx} + y \cdot \cot x = 2x + x^2 \cdot \cot x \right) \quad (x \neq 0)$

given that $y=0$ when $x = \frac{\pi}{2}$.

Ans.

$$\left(\frac{dy}{dx} + y \cot x = 2x + x^2 \cdot \cot x \right)$$

\downarrow
 \downarrow
 P
 Q

$$\frac{dy}{dx} + P \cdot y = Q$$

Linear Diff. Eqⁿ.

LDF

Integrating factor

$$I.F. = e^{\int P \cdot dx} \Rightarrow I.F. = e^{\int \cot x \cdot dx}$$

$$\Rightarrow I.F. = e^{\log \sin x} = \sin x \quad \checkmark$$

Solution: $y(I.F.) = \int Q(I.F.) \cdot dx + C$

$$\Rightarrow y \cdot (\sin x) = \int (2x + x^2 \cdot \cot x) \cdot \sin x \cdot dx + C$$

$$\Rightarrow y \cdot \sin x = \int (2x \sin x + x^2 \cdot \frac{\cos x}{\sin x} \cdot \sin x) \cdot dx + C$$

$$\Rightarrow y \sin x = \int (2x \sin x) \cdot dx + \int x^2 \cdot \cos x \cdot dx + C$$

$\swarrow \quad \searrow$
 ILATE

Int. by Parts.

$$\Rightarrow y \sin x = \int 2x \sin x + (x^2 \cdot \sin x - \int 2x \cdot \sin x \cdot dx) + C$$

$$\int I \cdot II = I \cdot \int II - \int (I' \cdot \int II) \Rightarrow y \sin x = x^2 \sin x + C$$

Solution. $y \sin x = x^2 \sin x + C$ ← General Solⁿ

$y=0, x=\frac{\pi}{2} \rightarrow 0 = \frac{\pi^2}{4} \cdot \sin \frac{\pi}{2} + C$

Particular solⁿ,

$\Rightarrow C = -\frac{\pi^2}{4}$

$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$

$\Rightarrow y = x^2 - \frac{\pi^2}{4 \sin x}$ ✓

e.g. Find the general solution of the differential equation $y \frac{dx}{dy} - (x + 2y^2) dy = 0$.

$\Rightarrow y \frac{dx}{dy} = (x + 2y^2) dy$

$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y$ $P = -\frac{1}{y}$ ✓ $Q = 2y$ ✓

Solution. $x(I.F) = \int Q \cdot (I.F) dy + C$

$\Rightarrow x\left(\frac{1}{y}\right) = \int (2y) \cdot \frac{1}{y} dy + C$

$\Rightarrow \frac{x}{y} = 2 \int dy + C$

$\Rightarrow \frac{x}{y} = 2y + C \Rightarrow x = 2y^2 + cy$ ✓

$\frac{dx}{dy} + P \cdot x = Q$

$I.F = e^{\int P \cdot dy}$
 $= e^{\int -\frac{1}{y} \cdot dy}$
 $= e^{-\log(y)} = e^{\log(y^{-1})}$
 $= y^{-1} = \left(\frac{1}{y}\right) = I.F.$

Exercise 9.6

 (Linear Differential Equations)

For each of the differential equations given, find the general solution —

Q.1 $\frac{dy}{dx} + 2y = \sin x$

$$\left[\frac{dy}{dx} + P \cdot y = Q \right] \text{ L.D.E}$$

Here $P = 2$

$Q = \sin x$

Integrating Factor

$$I.F = e^{\int P \cdot dx}$$

$$I.F = e^{\int 2 \cdot dx} = e^{2x}$$

Solution:

$$y(I.F) = \int [Q \times (I.F)] \cdot dx + C$$

$$\Rightarrow y \cdot e^{2x} = \int \sin x \cdot e^{2x} \cdot dx + C \quad \text{--- (I)}$$

Integration by Parts.

ILATE

$$(I) = \int \underbrace{\sin x}_I \cdot \underbrace{e^{2x}}_{II} \cdot dx$$

ILATE

$$\int I \cdot II = I \cdot \int II - \int (I' \cdot II) \cdot dx$$

$$I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} \cdot dx$$

ILATE

$$I = \frac{e^{2x} \cdot \sin x}{2} - \left[\cos x \cdot \frac{e^{2x}}{4} + \int \sin x \cdot \frac{e^{2x}}{4} \cdot dx \right]$$

$$I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} \int \sin x \cdot e^{2x} \cdot dx$$

$$\Rightarrow \textcircled{I} = \frac{e^{2x}}{4} \left\{ 2\sin x - \cos x \right\} - \left(\frac{I}{4} \right)$$

$$\Rightarrow \frac{I}{1} + \frac{I}{4} = \frac{e^{2x}}{4} \left\{ 2\sin x - \cos x \right\}$$

$$\Rightarrow \frac{5I}{4} = \frac{e^{2x}}{4} \left\{ 2\sin x - \cos x \right\}$$

$$\Rightarrow \boxed{I = \frac{e^{2x}}{5} (2\sin x - \cos x)} \quad \text{---} \textcircled{2}$$

By eqⁿ ① & ② $\Rightarrow \textcircled{y} \cdot e^{2x} = \frac{e^{2x}}{5} (2\sin x - \cos x) + C$

$$\Rightarrow y = \frac{\frac{e^{2x}}{5} (2\sin x - \cos x) + C}{e^{2x}} = \frac{1}{5} (2\sin x - \cos x) + C \cdot e^{-2x}$$

Q.2 $\frac{dy}{dx} + 3y = e^{-2x}$ L.D.E $\left[\frac{dy}{dx} + P \cdot y = Q \right]$
 $\boxed{P=3, Q=e^{-2x}}$ \downarrow \downarrow \textcircled{x} / Constant

$$\text{I.F.} = e^{\int P \cdot dx} = e^{\int 3 \cdot dx} = e^{3x} \quad \checkmark$$

Solution, $\boxed{y(\text{I.F.}) = \int \underline{Q}(\text{I.F.}) \cdot dx + C}$

$$\Rightarrow y \cdot e^{3x} = \int e^{-2x} \cdot e^{3x} \cdot dx + C$$

$$\Rightarrow y \cdot e^{3x} = \int e^x \cdot dx + C$$

$$\Rightarrow \boxed{y \cdot e^{3x} = e^x + C} \Rightarrow \boxed{y = e^{-2x} + C \cdot e^{-3x}} \quad \checkmark$$

Q.3 $\frac{dy}{dx} + \frac{y}{x} = x^2$

L.D.E

$\frac{dy}{dx} + P \cdot y = Q$

$P = \frac{1}{x}$, $Q = x^2$

$\frac{1}{x}$, x^2

$\log_a x = x$

I.F. = $e^{\int P \cdot dx} = e^{\int \frac{1}{x} \cdot dx} = e^{\log x} = x$

Solution:

$y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot dx + C$

$\Rightarrow y \cdot (x) = \int x^2 \cdot x \cdot dx + C$

$\Rightarrow xy = \int x^3 \cdot dx + C \Rightarrow xy = \frac{x^4}{4} + C$

Q.4 $\frac{dy}{dx} + (\sec x)y = \tan x$ ($0 \leq x < \frac{\pi}{2}$)

$\frac{dy}{dx} + P \cdot y = Q$ L.D.E.

I.F. = $e^{\int P \cdot dx}$
 $= e^{\int \sec x \cdot dx}$
 $= e^{\log [\sec x + \tan x]}$
 $= (\sec x + \tan x)$

Solution:

$y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot dx + C$

$\Rightarrow y (\sec x + \tan x) = \int \tan x \cdot (\sec x + \tan x) \cdot dx + C$

$\Rightarrow y (\sec x + \tan x) = \int (\sec x \tan x + \tan^2 x) \cdot dx + C$

$\Rightarrow y (\sec x + \tan x) = \int (\sec x \cdot \tan x + \sec^2 x - 1) \cdot dx + C$

$\Rightarrow y (\sec x + \tan x) = (\sec x + \tan x - x) + C$

Q.5 $\cos^2 x \cdot \left(\frac{dy}{dx}\right) + y = \tan x \quad \left(0 \leq x < \frac{\pi}{2}\right)$

$\cos^2 x$

L.D.E

$\Rightarrow \frac{dy}{dx} + y \cdot \sec^2 x = \tan x \cdot \sec^2 x$ // $\frac{dy}{dx} + P \cdot y = Q$

Divide

(x) diff terms.

$P = \sec^2 x$
 $Q = \tan x \cdot \sec^2 x$

I.F. = $e^{\int P \cdot dx}$
 I.F. = $e^{\int \sec^2 x \cdot dx}$
 I.F. = $e^{\tan x}$

Solution.

$y(I.F) = \int Q \cdot (I.F) \cdot dx + C$

$\Rightarrow y(e^{\tan x}) = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} \cdot dx + C$

$\Rightarrow y \cdot e^{\tan x} = \int e^{\tan x} \cdot \tan x \cdot \sec^2 x \cdot dx + C$

By substitution. $\tan x = t$
 $\sec^2 x \cdot dx = dt$ ✓

$\Rightarrow y \cdot e^{\tan x} = \int e^t \cdot t \cdot dt + C$

ILATE

(Integration by Parts)

$\int I \cdot II = I \cdot \int II - \int [I' \cdot \int II]$

$\Rightarrow y \cdot e^{\tan x} = t \cdot e^t - \int (1 \cdot e^t) \cdot dt + C$

$\Rightarrow y \cdot e^{\tan x} = (t \cdot e^t - e^t) + C$

$t = \tan x$

$\Rightarrow y \cdot e^{\tan x} = e^{\tan x} (\tan x - 1) + C$ ✓

Exercise 9.6

 (Linear Differential Equation)

Q.6 $x \frac{dy}{dx} + 2y = x^2 \cdot \log x$ ← Find the general Solution

x

L.D.E.

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = \underbrace{x \log x}_{Q}$$

$\frac{dy}{dx} + P \cdot y = Q$

$P = \frac{2}{x}$ $Q = x \log x$

Integrating Factor: I.F. = $e^{\int P \cdot dx} = e^{\int \frac{2}{x} \cdot dx}$

\Rightarrow I.F. = $e^{2 \log x} = e^{\log x^2} = x^2$

Solution: $y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \cdot dx + C$

$\Rightarrow y \cdot x^2 = \int x \log x \cdot x^2 \cdot dx + C$

$\Rightarrow y \cdot x^2 = \int x^3 \cdot \log x \cdot dx + C$

Integration by Parts

ILATE

$$\int I \cdot II = I \int II - \int (I' \cdot II)$$

$\Rightarrow y \cdot x^2 = \log x \cdot \frac{x^4}{4} - \int \left(\frac{1}{x} \cdot \frac{x^4}{4}\right) \cdot dx + C$

$\Rightarrow y \cdot x^2 = \frac{\log x \cdot x^4}{4} - \frac{1}{4} \int x^3 \cdot dx + C$

$\Rightarrow y \cdot x^2 = \frac{x^4 \cdot \log x}{4} - \frac{1}{4} \left(\frac{x^4}{4}\right) + C$

$$\Rightarrow (y)x^2 = \frac{x^4}{16} \{4 \log x - 1\} + C$$

$$\Rightarrow y = \frac{x^2}{16} \{4 \log x - 1\} + C \cdot x^{-2}$$

Q.7 $x \log x \cdot \frac{dy}{dx} + y = \frac{2}{x} \log x$

L.D.E

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x^2}$$

$$\left(\frac{dy}{dx} + P \cdot y = Q \right)$$

$$P = \frac{1}{x \log x}, \quad Q = \frac{2}{x^2}$$

$$\text{I.F} = e^{\int P \cdot dx} = e^{\int \frac{1}{x \log x} \cdot dx}$$

$$= e^{\log(\log x)}$$

$$= (\log x)$$

$$\int \frac{1}{x \log x} \cdot dx$$

$$\log x = t$$

$$\frac{1}{x} \cdot dx = dt$$

$$\int \frac{dt}{t} = \log t + C$$

$$= \log(\log x)$$

Solution: $y \cdot (\text{IF}) = \int Q \cdot (\text{IF}) \cdot dx + C$

$$\Rightarrow y \cdot \log x = \int \frac{2}{x^2} \cdot \log x \cdot dx + C$$

$$\Rightarrow y \cdot \log x = \log x \cdot \int \frac{2}{x^2} \cdot dx - \int \left[\left(\frac{1}{x} \right) \cdot \int \frac{2}{x^2} \cdot dx \right] \cdot dx + C$$

I L A T E
↑ ↑

$$\Rightarrow y \log x = \log x \cdot \left(\frac{-2}{x} \right) - \int \frac{1}{x} \cdot \left(\frac{-2}{x} \right) \cdot dx + C$$

$$\Rightarrow y \log x = \frac{-2 \log x}{x} + \int \frac{2}{x^2} \cdot dx + C$$

$$\rightarrow \left(\frac{-2}{x} \right)$$

$$\Rightarrow y \log x = -\frac{2 \log x}{x} + \int \frac{2}{x^2} \cdot dx + C$$

$\rightarrow -\frac{2}{x}$

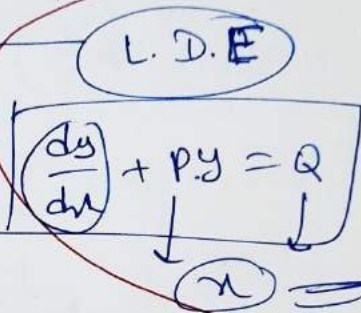
$$\Rightarrow y \log x = -\frac{2 \log x}{x} - \frac{2}{x} + C$$

$$\Rightarrow y \log x = -\frac{2}{x} \{ \log x + 1 \} + C$$

Q.8 $(1+x^2) \cdot \underline{dy} + 2xy \cdot \underline{dx} = \cot x \cdot \underline{dx}$ ($x \neq 0$)

\uparrow
 dx

$$\Rightarrow \frac{(1+x^2) dy + 2xy dx}{(1+x^2)} = \cot x$$



$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{\cot x}{1+x^2}$$

P Q

$$\frac{dy}{dx} + Py = Q$$

I. F. = $e^{\int P \cdot dx} = e^{\int \frac{2x}{1+x^2} \cdot dx} = e^{\log(1+x^2)} = (1+x^2)$

Solution: $y \cdot (IF) = \int Q(IF) \cdot dx + C$

$$\Rightarrow y \cdot (1+x^2) = \int \frac{\cot x}{(1+x^2)} \cdot (1+x^2) \cdot dx + C$$

$$\Rightarrow y \cdot (1+x^2) = \log |\sin x| + C$$

$$\Rightarrow y = \frac{\log |\sin x|}{1+x^2} + \frac{C}{1+x^2} = (1+x^2)^{-1} \cdot \log |\sin x| + C \cdot (1+x^2)^{-1}$$

Q.9 $x \frac{dy}{dx} + \frac{y}{x} - x + \frac{xy \cot x}{x} = 0, (x \neq 0)$

x

$$\frac{dy}{dx} + p \cdot y = Q$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} - 1 + \frac{y \cot x}{x} = 0$$

$$\Rightarrow \frac{dy}{dx} + y \left(\frac{1}{x} + \cot x \right) = 1$$

$\log m + \log n = \log mn$

$\log x + \log \sin x$

I.F. = $e^{\int p \cdot dx} = e^{\int \left(\frac{1}{x} + \cot x \right) \cdot dx} = e$

I.F. = $e^{\log_e(x \sin x)} = x \sin x$

Solution: $y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot dx + C$

$$\Rightarrow y \cdot (x \sin x) = \int 1 \cdot \frac{I}{x} \cdot \frac{II}{\sin x} \cdot dx + C$$

ILATE

Int. by Parts.

$$\Rightarrow y \cdot (x \sin x) = x \int \sin x \cdot dx - \int (1 \cdot (-\cos x)) \cdot dx + C$$

$$\Rightarrow y \cdot (x \sin x) = -x \cos x + \sin x + C$$

$$\Rightarrow y = \frac{-x \cos x}{x \sin x} + \frac{\sin x}{x \sin x} + \frac{C}{x \sin x}$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$$

Exercise 9.6 (Linear Differential Equations)

Find the general Solution

$$\boxed{\text{Q10}} \quad (x+y) \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x+y} \quad \Rightarrow \quad \frac{dx}{dy} = x+y$$

$$\Rightarrow \boxed{\frac{dx}{dy} - x = y}$$

$$\boxed{\frac{dx}{dy} + P \cdot x = Q}$$

$$\left. \begin{array}{l} P = -1 \\ Q = y \end{array} \right\}$$

Standard L. D. E.

$$\text{I.F.} = e^{\int P \cdot dy} = e^{-\int 1 \cdot dy}$$

(Integrating Factor)

$$= e^{-y} \quad \checkmark$$

Solution:

$$\boxed{x (\text{I.F.}) = \int Q (\text{I.F.}) \cdot dy + C}$$

$$\Rightarrow x \cdot e^{-y} = \int y \cdot e^{-y} \cdot dy + C$$

ILATE \leftarrow Int. by Parts

$$\int \text{II} = \int e^{-y} = \frac{e^{-y}}{-1}$$

$$\boxed{\int \text{I} \cdot \text{II} = \text{I} \int \text{II} - \int [\text{I}' \cdot \text{II}]}$$

$$\Rightarrow x e^{-y} = y (-e^{-y}) - \int [1 \cdot (-e^{-y})] \cdot dy + C$$

$$\Rightarrow x e^{-y} = -y \cdot e^{-y} + (-e^{-y}) + C$$

$$\Rightarrow x e^{-y} = -y e^{-y} - e^{-y} + C$$

$$\Rightarrow \boxed{x e^{-y} + y e^{-y} + e^{-y} = C}$$

$$\Rightarrow \boxed{(x+y+1) = C \cdot e^y} \quad \checkmark$$

Q11 $y dx + (x - y^2) dy = 0$

$$\frac{y dx + (x - y^2) dy}{y \cdot dy} = 0$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{x - y^2}{y} \right) = 0$$

L.D.E.
 $\frac{dx}{dy} + P \cdot x = Q$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y$$

$P = \frac{1}{y}, Q = y$

I.F. = $e^{\int P \cdot dy}$
 $= e^{\int \frac{1}{y} \cdot dy}$
 $= e^{\log y} = y$

$a^{\log x} = x$

Solution: $x(I.F) = \int Q \cdot (I.F) \cdot dy + c$

$$\Rightarrow x(y) = \int (y)(y) \cdot dy + c$$

$$\Rightarrow xy = \int y^2 \cdot dy + c \Rightarrow xy = \frac{y^3}{3} + c$$

$$\Rightarrow x = \frac{y^2}{3} + \frac{c}{y}$$

Q12 $(x + 3y^2) \left(\frac{dy}{dx} \right) = y \quad (y > 0)$

$$\Rightarrow (x + 3y^2) = y \cdot \frac{dx}{dy}$$

$$\Rightarrow \left(y \cdot \frac{dx}{dy} - x = 3y^2 \right) / y$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$$

$n \log m = \log m^n$

$$\frac{dx}{dy} + P \cdot x = Q$$

$P = -\frac{1}{y}, Q = 3y$

I.F. = $e^{\int P \cdot dy}$
 $= e^{-\int \frac{1}{y} \cdot dy}$
 $= e^{-\log y}$
 $= e^{\log(y^{-1})} = y^{-1} = \left(\frac{1}{y} \right)$

$$\boxed{x \text{ (I.F)} = \int Q \cdot \text{(I.F)} \cdot dy + C}$$

Solution.

$$x\left(\frac{1}{y}\right) = \int (3y) \cdot \left(\frac{1}{y}\right) \cdot dy + C$$

$$\Rightarrow \frac{x}{y} = \int 3 \cdot dy + C$$

$$\Rightarrow \frac{x}{y} = 3y + C$$

$$\Rightarrow \boxed{x = 3y^2 + Cy} \quad \checkmark$$

Exercise 9.6 (Linear Differential Equations)

Find a particular solution satisfying the given condition

Q.13 $\frac{dy}{dx} + 2y \tan x = \sin x$; $y=0$ when $x = \frac{\pi}{3}$

L.D.E

$$\frac{dy}{dx} + P \cdot y = Q$$

$P = 2 \tan x$ ✓

$Q = \sin x$ ✓

I.F. = $e^{\int P \cdot dx} = e^{\int 2 \tan x \cdot dx} = e^{2 \log \sec x}$

$a^{\log_a x} = x$

$\log m^n = n \cdot \log m$

$= e^{\log(\sec^2 x)}$

$= \sec^2 x = I.F.$

Solution:

$y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot dx + C$

$\Rightarrow y \cdot (\sec^2 x) = \int \sin x \cdot (\sec^2 x) \cdot dx + C$

$\Rightarrow y \cdot \sec^2 x = \int \sin x \cdot \frac{1}{\cos^2 x} \cdot dx + C$

$\Rightarrow y \cdot \sec^2 x = \int (\tan x \cdot \sec x) \cdot dx + C$

$y \cdot \sec^2 x = \sec x + C$

✓ $y=0, x = \frac{\pi}{3}$

$0 = \sec \frac{\pi}{3} + C$

$\Rightarrow 0 = 2 + C$
 $\Rightarrow C = -2$

$\frac{y}{\cos^2 x} = \frac{1}{\cos x} - 2$

$y = \cos x - 2 \cos^3 x$

Q.14 $(1+x^2) \frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$; $y=0$ when $x=1$.

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{1}{(1+x^2)^2}$$

L.D.E.

$$\frac{dy}{dx} + P y = Q$$

P Q

I.F. = $e^{\int P \cdot dx} = e^{\int \frac{2x}{1+x^2} \cdot dx} = e^{\log(1+x^2)} = (1+x^2)$

Solution: $y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot dx + C$

$$\Rightarrow y(1+x^2) = \int \frac{1}{(1+x^2)} \cdot (1+x^2) \cdot dx + C$$

$$\Rightarrow y(1+x^2) = \int \frac{1}{(1+x^2)} \cdot dx + C$$

$$\Rightarrow y(1+x^2) = \tan^{-1}(x) + C$$

$x=1, y=0$

Particular Solⁿ:

$$y(1+x^2) = \tan^{-1}(x) - \frac{\pi}{4}$$

$$0 \cdot () = \tan^{-1}(1) + C$$

$$\Rightarrow 0 = \frac{\pi}{4} + C$$

$$\Rightarrow C = -\frac{\pi}{4}$$

Q.15 $\frac{dy}{dx} - 3y \cot x = \sin 2x$; $y=2$ when $x = \frac{\pi}{2}$

$P = -3 \cot x$

$Q = \sin 2x$

L. D.E

$$\frac{dy}{dx} + P \cdot y = Q$$

I.F. = $e^{\int P \cdot dx} = e^{\int -3 \cot x \cdot dx} = e^{-3 \cdot \log(\sin x)}$

I.F. = $e^{\log(\sin x)^{-3}} = (\sin x)^{-3} = \frac{1}{\sin^3 x}$

Solⁿ

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \cdot dx + C$$

$$\Rightarrow y \cdot \left(\frac{1}{\sin^3 x}\right) = \int \sin 2x \cdot \frac{1}{\sin^3 x} \cdot dx + C$$

$$\Rightarrow \frac{y}{\sin^3 x} = \int \frac{2 \sin x \cdot \cos x}{\sin^3 x \sin^2 x} \cdot dx + C$$

$$\Rightarrow \frac{y}{\sin^3 x} = 2 \int (\cot x \cdot \operatorname{cosec} x) \cdot dx + C$$

$$\Rightarrow \frac{y}{\sin^3 x} = 2 (-\operatorname{cosec} x) + C$$

$y=2, x = \frac{\pi}{2}$

Particular Solⁿ

$$\Rightarrow \frac{y}{\sin^3 x} = \frac{-2}{\sin x} + C$$

$$\Rightarrow y = -2 \sin^2 x + 4 \sin^3 x$$

$$\frac{2}{\sin^3 \frac{\pi}{2}} = -2 \operatorname{cosec} \frac{\pi}{2} + C$$

$$\Rightarrow 2 = -2 + C$$

$$\Rightarrow \boxed{4 = C}$$

Q.16 Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

(+)

Ans. ATQ Slope of the tangent at point (x, y) = Sum of the coordinates of the point (x, y)

$\Rightarrow \frac{dy}{dx} = x + y$ ← D.E.

$\Rightarrow \frac{dy}{dx} - y = x$ ← L.D.E.

$\left. \begin{array}{l} P = -1 \\ Q = x \end{array} \right\}$

$\frac{dy}{dx} + P \cdot y = Q$

I.F. = $e^{\int P \cdot dx} = e^{-\int dx} = e^{-x}$

Solution: $y \cdot (I.F.) = \int Q(I.F.) \cdot dx + C$

$\Rightarrow y \cdot e^{-x} = \int x \cdot e^{-x} \cdot dx + C$

ILATE (Int. by Parts)

$\int I \cdot II = I \int II - \int (I' \cdot II)$

$\Rightarrow y \cdot e^{-x} = x(-e^{-x}) - \int 1 \cdot (-e^{-x}) \cdot dx + C$

$\Rightarrow y \cdot e^{-x} = -xe^{-x} + (-e^{-x}) + C$

$$\Rightarrow y \cdot e^{-x} = \underline{-x \cdot e^{-x} - e^{-x} + C}$$

$$\Rightarrow y \cdot e^{-x} + x e^{-x} + e^{-x} = C$$

$$\Rightarrow \boxed{e^{-x} (y + x + 1) = C} \rightarrow \text{Passes through } (0, 0)$$

$$\begin{aligned} & \text{①} \quad e^{-0} (0 + 0 + 1) = C \\ & \Rightarrow 1(1) = C \\ & \boxed{C = 1} \end{aligned}$$

$$\Rightarrow e^{-x} (y + x + 1) = 1$$

$$\Rightarrow \boxed{y + x + 1 = e^x}$$

Q.17 Find the equation of a curve passing through the point $(0, 2)$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Point on the curve = (x, y)

ATQ Sum of Coordinates of (x, y) = Slope of tangent + 5 at (x, y)

$$\Rightarrow x + y = \left(\frac{dy}{dx} \right) + 5$$

$$\Rightarrow x - 5 = \frac{dy}{dx} - y$$

$$\Rightarrow \boxed{\frac{dy}{dx} - y = \underline{x - 5}}$$

L. D. E.

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$$P = -1 \checkmark$$

$$Q = (x - 5) \checkmark$$

$$P = -1, Q = x-5$$

$$\text{I. F.} = e^{\int P \cdot dx} = e^{-\int dx} = e^{-x}$$

$$\text{Sol}^n \quad \boxed{y \cdot (\text{I. F.}) = \int Q \cdot (\text{I. F.}) \cdot dx + C}$$

$$\Rightarrow y \cdot (e^{-x}) = \int \underbrace{(x-5)}_I \cdot \underbrace{e^{-x}}_{II} \cdot dx + C$$

ILATE Int. by Parts

$$\Rightarrow y \cdot (e^{-x}) = (x-5) \cdot \int e^{-x} \cdot dx - \int [(1-0) \cdot \int e^{-x} \cdot dx] \cdot dx$$

$$\Rightarrow y \cdot (e^{-x}) = \underbrace{(x-5) \cdot (-e^{-x})} + \int e^{-x} \cdot dx + C$$

$$\Rightarrow y \cdot e^{-x} = \underbrace{-(x-5)e^{-x}} - \underbrace{e^{-x}} + C$$

$$\Rightarrow \boxed{y \cdot e^{-x} = [(5-x) \cdot -1] \cdot e^{-x} + C}$$

$$\Rightarrow y \cdot e^{-x} = [4-x] \cdot e^{-x} - 2$$

$$\Rightarrow \boxed{y = (4-x) - 2e^x}$$

Passes through (0, 2)

$$\begin{aligned} x &= 0 \\ y &= 2 \end{aligned}$$

$$2 \cdot e^{-0} = [4-0] \cdot e^{-0} + C$$

$$\Rightarrow 2 = 4 + C$$

$$\Rightarrow \boxed{-2 = C}$$

Q.18 The integrating factor of the differential eqⁿ. $x \frac{dy}{dx} - y = 2x^2$ is - (A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

$\Rightarrow \left(\frac{dy}{dx} - \frac{y}{x} = 2x \right)$ L.D.E. $\left(\frac{dy}{dx} + Py = Q \right)$
 \uparrow Q
 $P = -\frac{1}{x}$
I.F. $= e^{\int P \cdot dx} = e^{-\int \frac{1}{x} \cdot dx}$
 $= e^{-\log x} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$

Q.19 The integrating factor of the differential equation $(1-y^2) \left(\frac{dx}{dy} \right) + yx = ay$ ($-1 < y < 1$) is \rightarrow

- (A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Ans. $(1-y^2) \cdot \frac{dx}{dy} + yx = ay$ L.D.E.
 $\Rightarrow \left(\frac{dx}{dy} + \left(\frac{y}{1-y^2} \right) x = \left(\frac{ay}{1-y^2} \right) \right)$ $\left(\frac{dx}{dy} + P \cdot x = Q \right)$
 $\rightarrow Q$
 $P = \frac{y}{1-y^2}$ $(1-y^2)' = -2y$
I.F. $= e^{\int P \cdot dy} = e^{\int \frac{y}{1-y^2} \cdot dy} = e^{-\frac{1}{2} \int \frac{-2y}{1-y^2} dy}$
 $= e^{-\frac{1}{2} \log |1-y^2|} = e^{\log(1-y^2)^{-1/2}}$
 $= (1-y^2)^{-1/2} = \frac{1}{(1-y^2)^{1/2}} = \frac{1}{\sqrt{1-y^2}}$

Miscellaneous Exercise on Chapter (9)

Q.1 Indicate its order & degree (if defined).

(i) $\frac{d^2y}{dx^2} + 5x \left(\frac{dy}{dx}\right)^2 - 6y = \log x$ Polynomial in Derivatives

Highest order Derivative \rightarrow order of D.E. = 2 | Degree = 1 ✓

(ii) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$ Polynomial in Deri.

Highest order Derivative | order = 1 | Degree = 3

(iii) $\left(\frac{d^4y}{dx^4}\right) - \sin\left(\frac{d^3y}{dx^3}\right) = 0$

~~Not a~~ Polynomial in Derivatives

Highest order Derivative | order = 4 ✓ | Degree = Not Defined

Q.2 Verify that the given function (implicit or explicit) is a solution of the corresponding differential eqⁿ.

(i) $xy = \underbrace{ae^x + be^{-x}} + x^2$: $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy + x^2 - 2 = 0$

$a, b \rightarrow$ Arbitrary Constants

$\Rightarrow xy - x^2 = \underline{ae^x + be^{-x}}$ — (1)

diff. w.r.t. $(x) \rightarrow$

$\Rightarrow y + x \frac{dy}{dx} - 2x = ae^x - be^{-x}$ — (2)

Again by Diff. w.r.t. $(x) \rightarrow$

$\Rightarrow \left[\frac{dy}{dx} + \frac{dy}{dx} + x \cdot \left(\frac{d^2y}{dx^2} \right) - 2 = \underline{ae^x + be^{-x}} \right]$ — (3)

by eqⁿ (1) & (3) \Rightarrow

$\Rightarrow \left[x \cdot \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 2 = \underline{xy - x^2} \right]$

$\Rightarrow \left[x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy + x^2 - 2 = 0 \right]$

H.P.

$$(ii) \quad y = e^x (a \cos x + b \sin x) : \boxed{\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0}$$

(u.v)' ①

$$\frac{dy}{dx} = \underbrace{e^x (a \cos x + b \sin x)}_y + e^x (-a \sin x + b \cos x)$$

y (by eqn ①)

$$\frac{dy}{dx} = y + e^x (-a \sin x + b \cos x)$$

$$\Rightarrow \left(\frac{dy}{dx} \right) - y = \underline{e^x (-a \sin x + b \cos x)} \quad \text{--- ②}$$

By diff. w.r.t. (x) \rightarrow

$$\Rightarrow \frac{d^2 y}{dx^2} - \frac{dy}{dx} = \underbrace{e^x (-a \sin x + b \cos x)}_{\left(\frac{dy}{dx} - y \right) \text{ by eqn ②}} + e^x (-a \cos x - b \sin x)$$

$$\Rightarrow \frac{d^2 y}{dx^2} - \frac{dy}{dx} = \left(\frac{dy}{dx} - y \right) - \underbrace{e^x (a \cos x + b \sin x)}_y$$

$$\Rightarrow \frac{d^2 y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - (y)$$

$$\Rightarrow \boxed{\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0} \quad \checkmark$$

(iii) $y = x \sin 3x$: $\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

Particular solⁿ.

Free from Arbitrary Constants ✓

$y = x \sin 3x$ (u.v)

$\frac{dy}{dx} = 1 \cdot \sin 3x + 3x \cos 3x$

$\frac{d^2y}{dx^2} = 3\cos 3x + 3\cos 3x - 3x \sin 3x$ (3)

$\rightarrow = 6\cos 3x - 3x \sin 3x$

LHS = $\frac{d^2y}{dx^2} + 9y - 6\cos 3x$

$= (6\cos 3x - 3x \sin 3x) + 9(x \sin 3x) - 6\cos 3x$

$= 0 = \text{RHS.}$ ✓

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2) \frac{dy}{dx} - xy = 0$

diff. w.r.t (x) \rightarrow

$\Rightarrow 2x = 2y \cdot y' \cdot \log y + 2y^2 \cdot \left(\frac{1}{y}\right) \cdot y'$ $\Rightarrow x = y y' \left(2 \frac{x^2}{2y^2} + 1\right)$

$\Rightarrow x = 2y y' \cdot \log y + y \cdot y'$

$\Rightarrow x = y y' \{ 2 \log y + 1 \}$

$\therefore \log y = \frac{x^2}{2y^2}$

$\Rightarrow x = y y' \left(\frac{x^2 + y^2}{y^2}\right)$

$\Rightarrow xy = y' (x^2 + y^2)$

$\Rightarrow 0 = \frac{dy}{dx} (x^2 + y^2) - xy$

Miscellaneous Exercise on Chapter (9)

Q.3 Form the differential equation representing the family of curves given by $(x-a)^2 + 2y^2 = a^2$, where a is an arbitrary constant. Diff. Eqⁿ No. Arbi. Constants

Ans. $(x-a)^2 + 2y^2 = a^2$

$$\Rightarrow x^2 - 2ax + a^2 + 2y^2 = a^2$$

$$\Rightarrow x^2 + 2y^2 = 2ax$$

$$\Rightarrow \frac{(x^2 + 2y^2)}{x} = 2a$$

Diff. w.r.t. (x) $\left(\frac{u}{v}\right)' = \frac{u'v - u.v'}{v^2}$

$$\Rightarrow \frac{(2x + 4y.y') \cdot x - (x^2 + 2y^2) \cdot 1}{x^2} = 0$$

$$\Rightarrow 2x^2 + 4xyy' - (x^2 + 2y^2) = 0$$

$$\Rightarrow 4xy(y') = x^2 + 2y^2 - 2x^2$$

$$\Rightarrow \boxed{y' = \frac{2y^2 - x^2}{4xy}}$$

Q.4 Prove that $(x^2 - y^2) = c(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where c is a parameter.

Ans. $\frac{x^3 - 3xy^2}{y^3 - 3x^2y} = \frac{dy}{dx}$ \rightarrow H.D.E. Homog. Diff. eqⁿ
 \downarrow
 $\frac{y}{x} = v$

$$\frac{dy}{dx} = \frac{(x^3 - 3xy^2)/x^3}{(y^3 - 3x^2y)/x^3} = \frac{1 - 3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right)}$$

Substitution,

$$\frac{y}{x} = v$$

$$\Rightarrow y = xv$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - \frac{v}{1} = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$$

Variable separation

$$\Rightarrow \left(\frac{v^3 - 3v}{1 - v^4}\right) dv = \frac{dx}{x} \quad \text{integration}$$

$$\Rightarrow \int \left(\frac{v^3}{1 - v^4} - \frac{3v}{1 - v^4}\right) \cdot dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{-4} \int \frac{-4 \cdot v^3}{1 - v^4} dv - \frac{3}{2} \int \frac{2v \cdot dv}{1 - (v^2)^2} = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{4} \int \frac{dt}{t} - \frac{3}{2} \int \frac{dm}{1 - m^2} = \int \frac{dx}{x}$$

$$\Rightarrow \underbrace{-\frac{1}{4} \log(t)}_{\vdots} - \frac{3}{2} \left(\frac{1}{2} \log \left| \frac{1+m}{1-m} \right| \right) = \log x + \log C_1$$

$$\Rightarrow -\frac{1}{4} \left\{ \log(1 - v^4) + 3 \cdot \log \left| \frac{1+v^2}{1-v^2} \right| \right\} = \log |x C_1|$$

$$\Rightarrow \log(1-v^4) + \log\left(\frac{1+v^2}{1-v^2}\right)^3 = -4 \log |x c_1|$$

$$\Rightarrow \log\left\{\frac{(1-v^4) \cdot (1+v^2)^3}{(1-v^2)^3}\right\} = \log(x c_1)^{-4}$$

$$\Rightarrow \frac{\cancel{(1+v^2)}(1-v^2) \cdot (1+v^2)^3}{\cancel{(1-v^2)^3}(1-v^2)^2} = \frac{1}{(x c_1)^4}$$

$$1-v^4 = (1-v^2)(1+v^2)$$

\downarrow
 $(v^2)^2$

$$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = \frac{1}{(x c_1)^4}$$

Square root

$$\Rightarrow \frac{(1+v^2)^2}{(1-v^2)} = \frac{1}{(x c_1)^2}$$

$$v = \frac{y}{x}$$

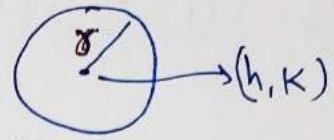
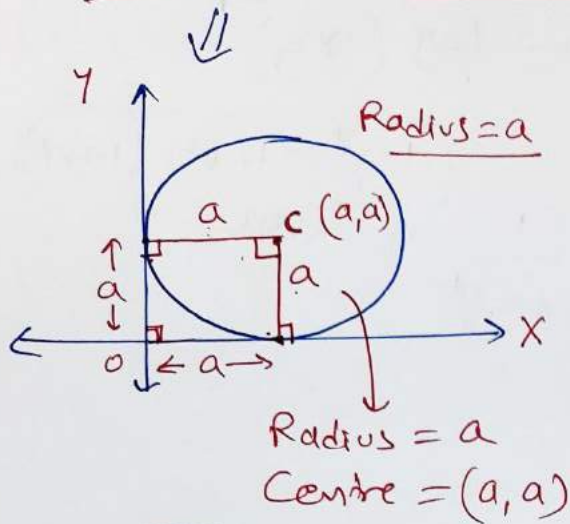
$$\Rightarrow \frac{\left(1 + \frac{y^2}{x^2}\right)^2}{1 - \frac{y^2}{x^2}} = \frac{1}{x^2 \cdot c_1^2} \Rightarrow \frac{\left(\frac{x^2+y^2}{x^2}\right)^2}{\left(\frac{x^2-y^2}{x^2}\right)} = \frac{1}{x^2 \cdot c_1^2}$$

$$\Rightarrow \frac{(x^2+y^2)^2}{\cancel{x^4} \cdot x^2} \times \frac{\cancel{x^2}}{(x^2-y^2)} = \frac{1}{x^2 \cdot c_1^2}$$

$$\Rightarrow c_1^2 \cdot (x^2+y^2)^2 = (x^2-y^2)$$

$$c \cdot (x^2+y^2)^2 = (x^2-y^2)$$

Q.5 Form the differential equation of the family of circles in the first quadrant which touch the coordinates axes.



Eqⁿ: $(x-h)^2 + (y-k)^2 = r^2$

order = 1 ← 1st order diff.

Eqⁿ: $(x-a)^2 + (y-a)^2 = a^2$

Family of curves

a = Arbitrary Constant.

Diff. Eqⁿ → Free from Arb. Const

$(x-a)^2 + (y-a)^2 = a^2$ — (1)
by Diff. w.r.t. (x)

$\Rightarrow \cancel{2}(x-a) + \cancel{2}(y-a) \cdot y' = 0$

$\Rightarrow x-a + yy' - ay' = 0$

$\Rightarrow x + yy' = a + ay'$

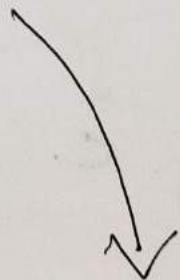
$\Rightarrow x + yy' = a(1+y')$

$\Rightarrow a = \left(\frac{x+yy'}{1+y'} \right)$ — (2)

by putting the value of 'a' from eqⁿ (2) to eqⁿ (1) →

$\Rightarrow \left(x - \frac{x+yy'}{1+y'} \right)^2 + \left(y - \frac{x+yy'}{1+y'} \right)^2 = \left(\frac{x+yy'}{1+y'} \right)^2$

$\Rightarrow \left(\frac{x+xy' - x - yy'}{1+y'} \right)^2 + \left(\frac{y+yy' - x - yy'}{1+y'} \right)^2 = \left(\frac{x+yy'}{1+y'} \right)^2$



$$\Rightarrow \left(\frac{xy' - yy'}{1+y'} \right)^2 + \left(\frac{y-x}{1+y'} \right)^2 = \left(\frac{x+yy'}{1+y'} \right)^2$$

$$\Rightarrow \underline{(y')^2 (x-y)^2} + \underline{(y-x)^2} = \underline{(x+yy')^2}$$

$$\boxed{\begin{array}{l} (a-b)^2 = (b-a)^2 \\ \underline{(4)^2 = 16}, \underline{(-4)^2 = 16} \end{array}}$$

$$\Rightarrow \underline{(y')^2 \cdot (x-y)^2} + \underline{(x-y)^2} = (x+yy')^2$$

$$\Rightarrow \boxed{(x-y)^2 ((y')^2 + 1) = (x+yy')^2} \quad \leftarrow$$

Miscellaneous Exercise on Chapter 9

Q.6 Find the general solution of the differential equation

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

Solⁿ

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Variable Separation →

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}} \quad (\text{integration})$$

$$\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}} \Rightarrow \sin^{-1}y = -\sin^{-1}x + C$$

$$\Rightarrow \boxed{\sin^{-1}y + \sin^{-1}x = C}$$

Q.7 Show that general solution of the differential equation

$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0 \text{ is given by } \underbrace{(x+y+1) = A(1-x-y-2xy)}_{\text{General Solution}},$$

where A is parameter.

Ans.

$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

variable separation

$$\Rightarrow \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

integration

$$\Rightarrow \int \frac{dy}{y^2+y+1} = -\int \frac{dx}{x^2+x+1}$$

$$I = \int \frac{dx}{x^2+x+1}$$

$$I = \int \frac{dx}{x^2+x+1} \quad (\text{Completing the square method})$$

$$I = \int \frac{dx}{x^2+x+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1}$$

$\underbrace{\hspace{10em}}_{a^2+2ab+b^2=(a+b)^2}$

$\frac{-1+4}{4} = \left(\frac{3}{4}\right) = \left(\frac{\sqrt{3}}{2}\right)^2$

$$I = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$\uparrow \quad \quad \quad \uparrow$
 $x \quad \quad \quad A$

$$\int \frac{dx}{A^2+x^2} = \frac{1}{A} \tan^{-1}\left(\frac{x}{A}\right) + c$$

$$I = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \cdot \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \quad \checkmark$$

Similar $\int \frac{dx}{y^2+y+1} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) \quad \checkmark$

Eqn. (1) $\rightarrow \int \frac{dy}{y^2+y+1} = - \int \frac{dx}{x^2+x+1}$

$$\Rightarrow \left(\frac{2}{\sqrt{3}}\right) \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = - \left(\frac{2}{\sqrt{3}}\right) \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \frac{2}{\sqrt{3}} \left\{ \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \right\} = C$$

$$\Rightarrow \tan^{-1}(A) + \tan^{-1}(B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\Rightarrow \left(\underbrace{\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)}_A + \underbrace{\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}_B \right) = \left(\frac{\sqrt{3}}{2} C \right)$$

by taking 'tan' to both sides.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \left(\frac{2y+1}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}} \right) = \tan\left(\frac{\sqrt{3}}{2} C\right)$$

$$\left(\frac{1}{1} - \left(\frac{2y+1}{\sqrt{3}}\right) \left(\frac{2x+1}{\sqrt{3}}\right) \right)$$

$$\Rightarrow \left(\frac{2x+2y+2}{\sqrt{3}} \right) = \tan\left(\frac{\sqrt{3}}{2} C\right)$$

$$\left(\frac{3 - (4xy + 2y + 2x + 1)}{2\sqrt{3}} \right)$$

$$\Rightarrow \sqrt{3} \left(\frac{2(x+y+1)}{2 - 4xy - 2y - 2x} \right) = \tan\left(\frac{\sqrt{3}}{2} C\right)$$

$$\Rightarrow \frac{2(x+y+1)}{2 - 4xy - 2y - 2x} = \frac{\tan\left(\frac{\sqrt{3}}{2} C\right)}{\sqrt{3}} = \text{New Name } \textcircled{A}$$

Parameters

$$\Rightarrow \boxed{(x+y+1) = A(1-2xy-y-x)}$$



[Q.8] Find the equation of the curve passing through the point $(0, \frac{\pi}{4})$ whose differential equation is

$$\sin x \cos y \cdot dx + \cos x \sin y \cdot dy = 0.$$

Ans. $\sin x \cdot \cos y \cdot dx + \cos x \cdot \sin y \cdot dy = 0$

$$\Rightarrow \sin x \cdot \cos y \cdot dx = - \cos x \cdot \sin y \cdot dy$$

V.S.

$$\Rightarrow \frac{\sin x \cdot dx}{\cos x} = - \frac{\sin y \cdot dy}{\cos y}$$

integration.

$$\Rightarrow \int \tan x \cdot dx = - \int \tan y \cdot dy$$

$$\Rightarrow \log(\sec x) = - \log(\sec y) + \log C$$

$$\Rightarrow \log(\sec x) + \log(\sec y) = \log C$$

$$\log m + \log n = \log mn$$

$$\Rightarrow \log(\sec x \cdot \sec y) = \log C$$

$$\Rightarrow \boxed{\sec x \cdot \sec y = C}$$

Curve passes through $(0, \frac{\pi}{4})$

$$\boxed{\sec x \cdot \sec y = \sqrt{2}}$$

$$\Rightarrow \frac{\sec x}{\sqrt{2}} = \frac{1}{\sec y}$$

$$\Rightarrow \boxed{\frac{\sec x}{\sqrt{2}} = \cos y}$$

$$\begin{aligned} \sec 0 \cdot \sec \frac{\pi}{4} &= C \\ \downarrow \\ 1 \cdot \sqrt{2} &= C \\ \boxed{C = \sqrt{2}} \end{aligned}$$

Q.9 Find the particular solution of the differential equation $(1+e^{2x})dy + (1+y^2)e^x dx = 0$; given that $y=1, x=0$.

Ans. $(1+e^{2x})dy = -(1+y^2)e^x dx$

variable separation.

by integration.

$$\Rightarrow \frac{dy}{1+y^2} = -\frac{e^x dx}{1+e^{2x}} \Rightarrow \int \frac{dy}{1+y^2} = -\int \frac{e^x dx}{1+(e^x)^2}$$

Formula

let $e^x = t$
 $e^x dx = dt$

$$\Rightarrow \tan^{-1} y = -\int \frac{dt}{1+t^2}$$

$$\Rightarrow \tan^{-1} y = -\tan^{-1}(t) + C$$

$$\Rightarrow \tan^{-1}(y) + \tan^{-1}(t) = C$$

$$\Rightarrow \tan^{-1}(y) + \tan^{-1}(e^x) = C$$

$y=1, x=0$ put. ?

$$\Rightarrow \tan^{-1}(1) + \tan^{-1}(e^0) = C$$

$$\Rightarrow \frac{\pi}{4} + \tan^{-1}(1) = C$$

$$\Rightarrow \frac{\pi}{2} = C$$

Particular solⁿ

$$\tan^{-1}(y) + \tan^{-1}(e^x) = \frac{\pi}{2}$$

Miscellaneous Exercise on Chapter 9 →

Q.10 Solve the differential equation $y \cdot e^{\frac{x}{y}} dx = (x e^{\frac{x}{y}} + y^2) dy$.

Ans. $\frac{dx}{dy} = \frac{x e^{\frac{x}{y}} + y^2}{y \cdot e^{\frac{x}{y}}}$

⇒ $\frac{dx}{dy} = \frac{x}{y} + \frac{y}{e^{\frac{x}{y}}}$

After Substitution.

⇒ $x + y \frac{dv}{dy} = x + \frac{y}{e^v}$

⇒ $y \frac{dv}{dy} = \frac{y}{e^v} \Rightarrow \frac{dv}{dy} = \frac{1}{e^v} \Rightarrow e^v \cdot dv = dy$

integration. ⇒ $\int e^v \cdot dv = \int dy$

⇒ $e^v = y + c$

⇒ $e^{\frac{x}{y}} = y + c$ ✓

$\frac{x}{y} = v$ Substitution

$x = y \cdot v$

$\frac{dx}{dy} = v + y \cdot \frac{dv}{dy}$

Variable Separation

Q.11 Find a particular solution of the differential equation $(x-y)(dx+dy) = (dx-dy)$ given that $y=-1$, when $x=0$.

Hint: put $(x-y=t)$.

Ans. $x-y=t$
diff. w.r.t. (x) ⇒ $(1 - \frac{dy}{dx}) = \frac{dy}{dx}$

⇒ $1 - \frac{dy}{dx} = \frac{dy}{dx}$

$\frac{(x-y)(dx+dy)}{dx} = \frac{dx-dy}{dx}$

⇒ $(x-y) \cdot (1 + \frac{dy}{dx}) = (1 - \frac{dy}{dx})$



$$\Rightarrow (x-y) \left(1 + \frac{dy}{dx}\right) = \left(1 - \frac{dy}{dx}\right)$$

Substitute $x-y=t$ ✓

$$\frac{dy}{dx} = 1 - \frac{dt}{dx}$$

$$\Rightarrow (t) \left(1 + 1 - \frac{dt}{dx}\right) = 1 - \left(1 - \frac{dt}{dx}\right)$$

$$\Rightarrow \underbrace{2t - t \cdot \frac{dt}{dx}} = \underbrace{x - x + \frac{dt}{dx}} \Rightarrow \underbrace{2t = t \frac{dt}{dx} + \frac{dt}{dx}}$$

$$\Rightarrow 2t = (t+1) \frac{dt}{dx} \Rightarrow dx = \left(\frac{t+1}{2t}\right) dt$$

integration: $\int dx = \int \left(\frac{1}{2} + \frac{1}{2t}\right) \cdot dt$

$$\Rightarrow x = \frac{t}{2} + \frac{1}{2} \cdot \log|t| + c$$

$$\Rightarrow \boxed{x = \frac{x-y}{2} + \frac{1}{2} \log|x-y| + c}$$

$y = -1, x = 0$ put

$$\Rightarrow 0 = \frac{1}{2} + \frac{1}{2} \log|1| + c \Rightarrow \boxed{c = -\frac{1}{2}}$$

Particular solⁿ: $x = \frac{x-y}{2} + \frac{1}{2} \log|x-y| - \frac{1}{2}$

$$\Rightarrow \frac{x}{1} - \left(\frac{x-y}{2}\right) + \frac{1}{2} = \frac{1}{2} \log|x-y|$$

$$\Rightarrow \frac{2x - x + y + 1}{2} = \frac{1}{2} \log|x-y|$$

$$\Rightarrow \boxed{(x+y+1) = \log|x-y|}$$

Q.12 Solve the differential eqⁿ. $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \right] \frac{dx}{dy} = 1 \quad (x \neq 0)$

L. D. F.

$\frac{dy}{dx} + P \cdot y = Q$
 Standard form

$$\Rightarrow \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\Rightarrow \frac{e^{-2\sqrt{x}}}{\sqrt{x}} = \frac{dy}{dx} + \frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$P = \frac{1}{\sqrt{x}}$
 $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Integrating factor

$$\begin{aligned} \text{I.F.} &= e^{\int P \cdot dx} = e^{\int \frac{1}{\sqrt{x}} \cdot dx} = e^{\int x^{-1/2} \cdot dx} \\ &= e^{\left(\frac{x^{-1/2+1}}{-1/2+1} \right)} = e^{\left(\frac{x^{1/2}}{1/2} \right)} = e^{2\sqrt{x}} = \text{I.F.} \end{aligned}$$

Solution:

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \cdot dx + C$$

$$\Rightarrow y \cdot (e^{2\sqrt{x}}) = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} \cdot dx + C$$

$$\Rightarrow y \cdot (e^{2\sqrt{x}}) = \int \frac{1}{\sqrt{x}} \cdot dx + C$$

$$\Rightarrow y \cdot (e^{2\sqrt{x}}) = 2\sqrt{x} + C$$

Q.13 Find a particular solution of the diff. eqⁿ.

$$\frac{dy}{dx} + y \cot x = \underline{4x \operatorname{cosec} x}, \quad (x \neq 0), \quad \text{given that } \underline{y=0, x=\frac{\pi}{2}}$$

L.D.E. $\frac{dy}{dx} + P \cdot y = Q$

$$P = \cot x$$

$$Q = 4x \operatorname{cosec} x$$

$$\text{I.F.} = e^{\int P \cdot dx}$$

$$= e^{\int \cot x \cdot dx} = e^{\log |\sin x|} = \sin x$$

Solution: $y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \cdot dx + C$

$$\Rightarrow y \cdot \sin x = \int 4x \operatorname{cosec} x \cdot \sin x \cdot dx + C$$

$$\Rightarrow y \cdot \sin x = 2x^2 + C$$

$$\Rightarrow \boxed{y \sin x = 2x^2 + C} \quad \text{General solⁿ}$$

$$y=0, x=\frac{\pi}{2}$$

$$\Rightarrow 0 \cdot \left(\sin \frac{\pi}{2}\right) = 2 \left(\frac{\pi}{2}\right)^2 + C$$

$$\Rightarrow -2 \cdot \frac{\pi^2}{4} = C$$

$$\Rightarrow \boxed{-\frac{\pi^2}{2} = C}$$

Particular Solⁿ.

$$\boxed{y \sin x = 2x^2 - \frac{\pi^2}{2}}$$

Miscellaneous Exercise on Chapter 9

Q.14 Find a particular solution of the diff. eqⁿ.

$$(x+1) \frac{dy}{dx} = 2e^{-y} - 1, \text{ given that } \underline{y=0 \text{ when } x=0.}$$

Ans: $(x+1) \frac{dy}{dx} = (2e^{-y} - 1)$

- ① Variable Separation ✓
- ② Homo. Diff. Eqⁿ.
- ③ Linear Diff. Eqⁿ.

$$\Rightarrow \frac{dy}{(2e^{-y} - 1)} = \frac{dx}{(x+1)}$$

integration

$$\Rightarrow \int \frac{dy}{\left(\frac{2}{e^y} - 1\right)} = \int \frac{dx}{(x+1)} \Rightarrow \int \left(\frac{dy}{\frac{2 - e^y}{e^y}}\right) = \int \frac{dx}{x+1}$$

$$\Rightarrow - \int \frac{-e^y \cdot dy}{(2 - e^y)} = \int \frac{dx}{x+1}$$

log mn
" "
log m + log n

$$\Rightarrow - \int \frac{dt}{t} = \int \frac{dx}{x+1} \Rightarrow - \log|t| = \log|x+1| + \log c$$

$$\Rightarrow - \log(2 - e^y) = \log[(x+1) \cdot c]$$

$n \cdot \log m = \log m^n$

$$\Rightarrow \log(2 - e^y)^{-1} = \log[c \cdot (x+1)]$$

$$\Rightarrow \frac{1}{2 - e^y} = c \cdot (x+1)$$

$y=0, x=0$

$$\Rightarrow \frac{1}{2 - e^0} = c \cdot (1)$$

$$\Rightarrow \frac{1}{2-1} = c$$

$$\Rightarrow \boxed{c=1}$$

Particular solution

$$\frac{1}{2 - e^y} = C(x+1) \Rightarrow \frac{1}{x+1} = 2 - e^y$$

$C=1$

$$\Rightarrow e^y = 2 - \frac{1}{x+1} \Rightarrow e^y = \frac{2x+2-1}{x+1}$$

$$\Rightarrow e^y = \frac{2x+1}{x+1}$$

$$\Rightarrow y = \log \left(\frac{2x+1}{x+1} \right)$$

Exponential fn. \leftrightarrow log

Q.15 The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999, and 25000 in the year 2004, what will be the population of the village in 2009?

Ans.

	(+) Time Duration	Time line Years	Population
Case - I	t=0	1999	P=20000
Case - II	t=5	2004	P=25000
Case - III	t=10	2009	P=?

Rate of change of population = $\frac{dp}{dt}$

Let 'P' be populat. of village at some time

ATQ. $\frac{dp}{dt} \propto P$

$$\Rightarrow \frac{dp}{dt} = K.P$$

K = ~~P~~ proportionality Constant

Diff. eqn.

$$\Rightarrow \boxed{\frac{dp}{dt} = kP}$$

by variable separation:

$$\Rightarrow \frac{dp}{P} = k dt$$

integration

$$\Rightarrow \int \frac{dp}{P} = k \int dt \Rightarrow \boxed{\log(P) = kt + C}$$

Const. Const.

Case-I $t=0, P=20000$

$$\Rightarrow \log 20000 = k(0) + C$$

$$\Rightarrow \boxed{C = \log 20000} \quad \checkmark$$

updated: $\boxed{\log P = kt + \log 20000}$

Case-II $t=5, P=25000$

$$\Rightarrow \log(25000) = k(5) + \log(20000)$$

$$\Rightarrow \log\left(\frac{5}{4}\right) = 5k \Rightarrow \boxed{k = \frac{1}{5} \log\left(\frac{5}{4}\right)}$$

updated $\boxed{\log P = \frac{1}{5} \log\left(\frac{5}{4}\right) \cdot t + \log 20000}$

$t = 10$ years (A in 2009)

$$\Rightarrow \log(P) = \frac{1}{5} \log\left(\frac{5}{4}\right) \cdot 10^2 + \log(20000)$$

$$\Rightarrow \log P = 2 \cdot \log\left(\frac{5}{4}\right) + \log 20000$$

$$\Rightarrow \log P = \log\left(\frac{5}{4}\right)^2 + \log 20000$$

$$\Rightarrow \log P = \log \left[\frac{5 \times 5 \times 20000}{4 \times 4} \right]$$

$$\Rightarrow \boxed{P = 31250}$$

Q.16 The general solution of the diff. eqⁿ.

$\frac{y dx - x dy}{y} = 0$ is - (A) $xy = c$ (B) $x = cy^2$
 (C) $y = cx$ (D) $y = cx^2$

$\Rightarrow y dx - x dy = 0$

$\Rightarrow y \cdot dx = x \cdot dy$

variable separation,

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$

Int. $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\Rightarrow \log x = \log y + \log C_1$

$\Rightarrow \log x = \log(y \cdot C_1)$

$\Rightarrow x = y \cdot C_1$

$\Rightarrow y = \frac{x}{C_1} \Rightarrow y = cx$

Q.17 \rightarrow Directly in Book

Q.18 The general solution of the differential eqⁿ.

$e^x dy + (ye^x + 2x) dx = 0$ is - (A) $xey + x^2 = c$

(B) $xey + y^2 = c$

(C) $ye^x + x^2 = c$

(D) $ye^y + x^2 = c$

L.D.E. $\left(\frac{dy}{dx} + P \cdot y = Q\right)$

$\Rightarrow \frac{e^x dy + (ye^x + 2x) dx}{dx} = 0$

$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$

$\Rightarrow \frac{dy}{dx} + 1 \cdot y = \frac{-2x}{e^x}$

$P=1, Q = \frac{-2x}{e^x}$

I.F. = $e^{\int P \cdot dx}$
 $= e^{\int 1 \cdot dx} = e^x$

Solution,

$y \cdot (I.F.) = \int Q(I.F.) \cdot dx + C$

$\Rightarrow y(e^x) = \int \frac{-2x}{e^x} \cdot e^x \cdot dx + C$

$\Rightarrow y \cdot e^x = -x^2 + C$

$\Rightarrow y \cdot e^x + x^2 = C$